FIRST LESSONS

IN

PLANE GEOMETRY.

TOGETHER

WITH AN APPLICATION OF THEM

TO THE

SOLUTION OF PROBLEMS.

SIMPLIFIED FOR BOYS NOT VERSED IN ALGEBRA.

BY FRANCIS J. GRUND,
Teacher of Mathematics, at Chauncy Hall-School.

BOSTON:
PUBLISHED BY CARTER AND HENDEE.
MDCCCXXX.

PRINTED BY I. R. BUTTS....BOSTON.

TABLE OF CONTENTS.

SECTION III.

SECTION IV.

SECTION V.

Application of the foregoing Principles to the solution of Geometrical Problems.

PART I.

PART II.

PART III.

PART IV.

GEOMETRY.

INTRODUCTION.

If, without regarding the qualities of bodies, viz: their smoothness, roughness, colour, compactness, tenacity, &c., we merely consider *the space which they fill—their extension in space*—they become the special subject of mathematical investigation, and the science which treats of them, is called *Geometry*.

The extensions of bodies are called *dimensions*. Every body has three dimensions, viz: *length*, *breadth*, and *depth*.* In speaking of a wall or a house, for instance, you can form no idea of it without conceiving it to extend in length, breadth, and

* The term *height*, is sometimes used for breadth. But height and breadth express originally the same thing. When a surface is placed horizontally, we speak of its length and *breadth*; and when perpendicularly, of its length and *height*.

depth; and the same is the case with every other body you can think of.

The limits or confines of bodies are called surfaces (superficies), and may be considered independently of the bodies themselves. So you may look at the front of a house, and inquire how long and how high is that house, without regarding its depth; or the length and breadth of a field, without asking how deep it goes into the ground, &c. In all such cases, you merely consider *two* dimensions. *A surface is, therefore, defined to be an extension in length and breadth without depth.*

The limits or edges of surfaces are called *lines*, and may again be considered independently of the surfaces themselves. You may ask, for instance, how *long* is the front of such a house, without regarding its height; or how far is it from Boston to Roxbury, without inquiring how *broad* is the road? Here, you consider evidently only *one* dimension; and *a line, therefore, is defined to be an extension in length without breadth or depth.*

The beginning and end of lines are called *points*. They merely mark the *positions*

of lines, and can, therefore, of themselves, have no magnitude. To give an example: when you set out from Boston to Roxbury, you may indicate the place you start from, which you may call the *point* of starting. If this chances to be Marlborough Hotel, you do not ask how *long*, or *broad*, or *deep* that place is; it suffices for you to know the *spot* where you begin your journey. A point is therefore defined to be *mere position, without either length or breadth.*

Remark. A point is *represented* on paper or on a board, by a small dot. A line is *drawn* on paper with a pointed lead pencil or pen; and on the board, with a thin mark made with chalk. The extensions of surfaces are *indicated* by lines; and bodies are *represented* on paper or on the board, according to the rules of perspective.

* *
*

In order to begin the study of Geometry, it is necessary, first, to acquaint ourselves with the meaning of some terms, which are frequently made use of in books treating on that science.

Definitions.

A *line* is called *straight*, when every part of it lies in *the same direction*, thus,

Any line in which no part is *straight*, is called a *curve* line.

A *geometrical plane* is a surface, in which two points being taken at pleasure, the straight line joining them will lie entirely in that surface.* A surface in which no part is plane, is called a *curved* surface. Any plane surface, terminated by lines, is called a *geometrical figure.*

The simplest geometrical figure, terminated by *three* straight lines, is called a *triangle.*

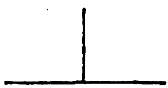
A geometrical figure, terminated by *four* straight lines, is called a *quadrilateral*, by 5, a *pentagon*—by 6, an *hexagon*—by 7, an

* The teacher can give an illustration of this definition, by taking, any where on a piece of paste board, two points and joining them by a piece of stiff wire. Then, by bending the board, the wire which represents the line will be off the board, and you have a curved surface; and by stretching the board, so as to make the wire fall upon it, you have a plane.

heptagon—by 8, an *octagon*—by 9, a *nonagon*—by 10, a *decagon*, &c.

Any geometrical figure, terminated by more than three straight lines, is (by some authors) called a *polygon*.*

When two straight lines meet, they form an *angle;* the point at which they meet is called the *vertex*, and the lines themselves are called the *legs* of the angle. When a straight line meets another, so as to make the two adjacent angles equal, the angles are called *right* angles, and the lines are said to be *perpendicular* to each other,



Any angle smaller than a right angle, is called *acute*,

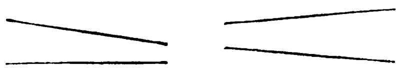
and when greater than a right angle, an *obtuse* angle.†

* Legendre calls all geometrical figures polygons.

† Angles are measured by an arc of a circle, described

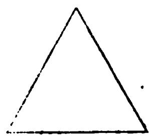
Two lines, which, lying in the same plane, and however so far extended in both directions, never meet, are said to be parallel to each other.

When two lines, situated in the same plane, are not parallel, they are either *converging* or *diverging*. Two lines are said to be converging, if, when extended, in the direction we consider, they grow nearer each other, and diverging, if the reverse takes place.



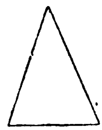
Converging. Diverging.

A triangle is called *equilateral*, when all its sides are equal.



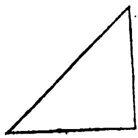
with any radius between its legs. Here the teacher may state, that the circle is divided into 360 equal parts, called degrees; each degree, again, into 60 equal parts, called minutes; a minute, again, subdivided into 60 equal parts, called seconds, &c.; and that the magnitude of an angle can thus be expressed in degrees, minutes, seconds, &c. of an arc of a circle, contained between its legs.

A triangle is called *isosceles* when *two* of its sides only are equal.



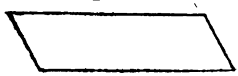
A triangle is called *scalene*, when none of its sides are equal.

A triangle is also called *right-angular*, when it contains a *right angle*;

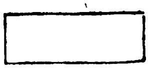


and *oblique-angular* when it contains no right angle.

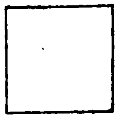
A *parallelogram* is a quadrilateral whose opposite sides are parallel.



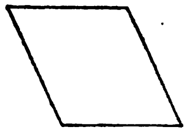
A *rectangle* or *oblong* is a right angular parallelogram.



A *square* is a rectangle whose sides are all equal.



A *rhombus* or *lozenge* is a *parallelogram* whose sides are all equal.

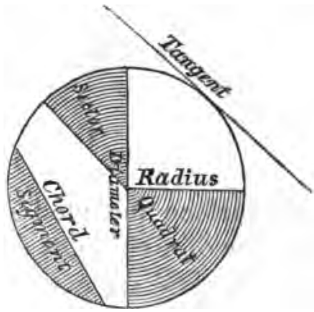


A *trapezoid* is a quadrilateral in which two sides only are parallel.

A straight line joining two verteces, which are not on the same side of a geometrical figure, is called a *diagonal.*

The simplest of all geometrical figures, is that which is terminated by one reentering curve line, all points of which are at an equal distance from one and the same point called the *centre.* A plane

surface thus terminated, is called a *circle*.

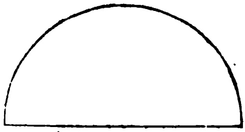


The curve line itself is called the *circumference*. Any part of it is called an *arc*. A straight line drawn from the centre of a circle, to any point of the circumference, is called a *radius*. A straight line, drawn from one point of the circumference to the other, passing through the centre, is called a *diameter*. A straight line joining any two points of the circumference, without passing through the centre, is called a *chord*.

The plane surface included within an arc of a circle and the chord on which it stands, it is called a *segment*.

The arc of a circle which stands on a diameter is called a *semi-circumference*. A plane surface included within a semi-cir-

cumference and a diameter, is called a *semicircle*.



The plane surface included within two radii and an arc of a circle, is called a *sector*. If the two radii are perpendicular to each other, the sector, is called a *quadrant*.

A straight line, which, drawn without the circle, and however so far extended in both directions, meets the circumference only in one point, is called a *tangent*. (See the figure on page 9.)

QUESTIONS ON DEFINITIONS.

What is that science called, which treats of the extensions of bodies, considered separately from all their other qualities?

What are the extensions of bodies called?

What are the limits or confines of bodies called?

How do you *define* a surface?

What are the limits of surfaces called?

How do you *define* a line?

What are the beginning and end of lines called?

How do you *define* a point?

How is a geometrical point *represented?*

How is a line represented? How a surface?

How do you define a *straight* line?

What do you call a line, in which no part is straight?

What is that surface called, in which, when two points are taken at pleasure, the straight line joining them will lie entirely within it?

What do you call a surface, in which no part is plane?

What is a plane surface called when terminated by lines?

By how many straight lines is the simplest rectilinear figure terminated?

What do you call it?

What do you call a geometrical figure terminated by four straight lines?

What, if terminated by five straight lines?

What, if by six? By seven? By eight? By nine? By ten?

What are all geometrical figures, terminated by more than three straight lines, called?

When two straight lines meet, what do they form?

What is the point, where the lines meet, called?

If one straight line meets another, so as to make the two adjacent angles equal, what do you call these angles?

What do you call the lines themselves?

What is an angle, which is smaller than a right angle, called?

What, an angle larger than a right angle?

What do you call two lines, which, situated in the same plane, and however so far extended both ways, never meet?

When are two lines said to be converging? When, diverging?

When a triangle has all its sides equal, what is it called?

When two of its sides only are equal, what?

When none of its sides are equal, what?

What is a triangle called, when it contains a right angle?

What, if it does not contain one?

What is a quadrilateral, whose opposite sides are equal, called?

What is a right-angular parallelogram called?

What is an equilateral rectangle called?

What, an equilateral *parallelogram?*

What, an equilateral in which two sides only are parallel?

What is the simplest of all geometrical figures called?

How is a circle terminated?

What is the line called which terminates a circle?

What is any part of the circumference called?

What, a straight line, drawn from the centre, to any part of the circumference?

What, a straight line joining two points of the circumference, and passing through the centre?

What, a straight line joining two points of the circumference, without passing through the centre?

What is the plane surface, included within an arc and the chord which joins the two extremities, called?

What is that part of the circumference called, which is cut off by the diameter?

What, the plane surface within a semicircumference and a diameter?

What, the surface within an arc of a circle and the two radii drawn to its extremities?

What is the sector called, if the two radii are perpendicular to each other?

What is the name of a straight line, drawn without the circle, which, extended both ways ever so far, touches the circumference only in one point?

NOTATION AND SIGNIFICATIONS.

For the sake of shortening expressions, and thereby to facilitate language, mathematicians have agreed to adopt the following signs:

= stands for equal; *e. g.*, the line AB=CD, means, that the line AB is equal to the line CD.

+ stands for *plus* or *more*; *e. g.*, the lines AB+CD, means, that the length of the line CD is to be added to the line AB.

— stands for *minus* or *less*; *e. g.*, line AB—CD means that the length of the line

CD is to be taken away from the line AB.

$\times$ is the sign of Multiplication.

: is the sign of Division.

$<$ stands for *less than; e. g.*, the line $AB < CD$ means that the line AB is shorter than the line CD.

$>$ stands for *greater than; e. g.*, the line $AB > CD$ means that the line AB is longer than the line CD.

$\parallel$ stands for parallel; *e. g.*, the line $AB \parallel CD$ means that the line AB is parallel to the line CD.

⋕ stands for equal and parallel; *e. g.*, the line AB⋕CD means that the line AB is equal, and, at the same time, parallel to the line CD.

A point is denoted by a single letter of the alphabet chosen at pleasure; *e. g.*,

·B

the point B.

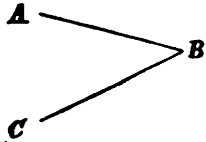
A line is represented by two letters placed at the beginning and end of it; *e. g.*,

A ———————— *B*

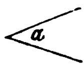
the line AB.

An angle is commonly denoted by three

letters, the one that stands at the vertex always placed in the middle; *e. g.*,

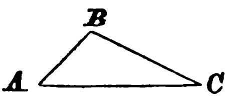


the angle ABC or CBA. It is sometimes also represented by a single letter placed within the angle; *e. g.*,



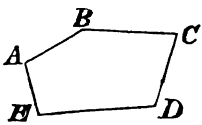
the angle *a*.

A triangle is denoted by three letters, placed at the three verteces; *e. g.*,



the triangle ABC.

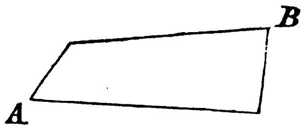
Any polygon is denoted by as many letters as there are verteces; *e. g.*,



the pentagon ABCDE.

A quadrilateral is something also denot-

ed only by two letters, placed at the opposite verteces ; *e. g.*



the quadrilateral AB.

QUESTIONS ON NOTATION AND SIGNIFICATIONS.

What signs do mathematicians use to abreviate writing?

What is the sign of equality?

What sign stands for plus, or more?

What for minus, or less?

What for multiplication?

What for division?

What for *less than*?

What for *more than*?

What for parallel?

What for equal and parallel?

How is a point denoted?

How a line?

How an angle?

How a triangle?

How a quadrilateral?

How any polygon?

Axioms.

There are certain invariable truths, which are at once plain and evident to every mind, and which are frequently made use of, in the course of geometrical reasoning. As you will frequently be obliged to refer to them, it will be well to recollect the following ones particularly:

TRUTH I.

Things, which are equal to the same thing, are equal to one another.

TRUTH II.

Things, which are similar to the same thing, are similar to one another.

TRUTH III.

If equals be added to equals, the wholes are equal.

TRUTH IV.

If equals be taken from equals, the remainders are equal.

TRUTH V.

The whole is greater than any one of its parts.

TRUTH VI.

The sum of all the parts is equal to the whole.

TRUTH VII.

Magnitudes, which coincide with one another, that is, which exactly fill the same space, are equal to one another.

TRUTH VIII.

Between two points only one straight line can be drawn.

TRUTH IX.

The straight line is the shortest way from one point to another.

TRUTH X.

Through one point, without a straight line, only one line can be drawn parallel to that same straight line.

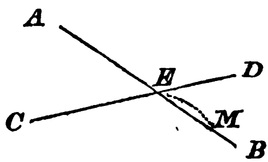
SECTION I.

OF STRAIGHT LINES AND ANGLES.

QUERY I.

In how many points can two straight lines cut each other?

Answer. In *one* only.



Q. But could not the two straight lines AB, CD, which cut each other in the point E, have another point common; that is, could not a part of the line ED bend over and touch the line AB in M?

A. No.

Q. Why not?

A. Because there would be *two* straight lines, drawn between the same points E and M, which is impossible. (Truth 8.)

QUERY II.

If two lines have any part common, what must necessarily follow?

A. They must coincide with each other throughout, and make but one and the same straight line.

Q. How can you 

prove this, for instance, of the two lines CA, MB, which have the part MA common?

A. Because, between the two points A and M they cannot vary; otherwise there would be more than one straight line drawn between the two points M and A.

Q. But is it not possible for either of the parts MC or AB to vary from the direction AM?

A. No.

Q. Why not?

A. Because MA forming part of the line AB as well as of the line AC; MC and AB are but the continuations of the same straight line AM.

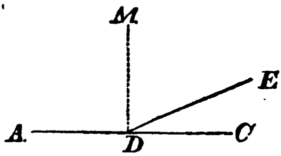
Q. Will you now repeat the whole of your reasoning, and prove that two straight lines cannot have any part common, without coinciding with each other throughout, and making but one and the same straight line?

* * *

QUERY III.

If a straight line meets another, how great will be the sum of the two adjacent angles, which it makes with it, taking a right angle for the measure?

A. It will be equal to two right angles.



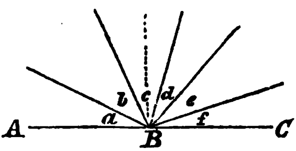
Q. How do you prove this of the two angles, formed by the line ED, meeting the line AC in the point D ?

A. Because, if in D you erect the perpendicular DM, the two angles ADE and CDE will occupy exactly the same space as the two right angles, ADM and CDM, formed by the meeting of the perpendicular; namely, all the space on one side of the line AC.

Q. Can you prove the same of the sum of the two adjacent angles, formed by the meeting of any other two straight lines ?

* * *

QUERY IV.



What will be the sum of any number of angles, a, b, c, d, e, &c., formed at the same point, and on the same side of the straight line AC, taking again a right angle for the measure?

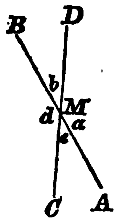
A. It will also be equal to two right angles.

Q. Why ?

A. Because, by erecting in that point B a perpendicular to AC, all these angles will be found to occupy the same space as the two right angles, made by the perpendicular.

QUERY V.

When two straight lines, AB, CD, cut each other, what relation will the angles which are opposite to each other at the vertex M, bear to each other?

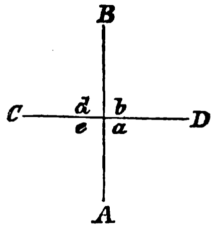


A. They will be equal to each other.

Q. How can you prove it?

A. Because, if you add the same angle *a*, first to *b*, and then to *e*, the sum will, in both cases, be the same (equal to two right angles); and if the angle *b* were not equal to the angle *e*, this could not be; and in the same manner I can prove that the two angles *a* and *d* are equal.

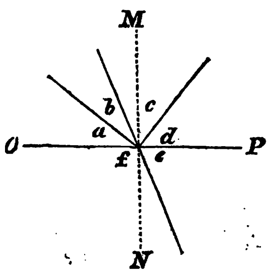
Q. If the lines CD, AB, are perpendicular to each other, what remark can you make in relation to the angles *d*, *b*, *e*, *a*?



A. That each of these angles will be a right angle.

Q, And what is the sum of all the angles, *a*, *b*, *c*, *d*, *e*, *f*, around the same point, equal to?

A. To four right angles.



Q. Why?

A. Because, if through

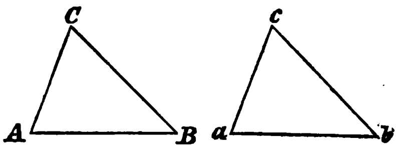
that point, you draw a perpendicular to any of the lines, for instance the perpendicular MN to the line OP, all the angles, *a*, *b*, *c*, *d*, *e*, *f*, taken together, occupy the same space which is occupied by the four right angles, formed by the intersection of the two perpendiculars *MN*, *OP*.

QUERY VI.

If a triangle has one side, and the two adjacent angles, equal to one side and the two adjacent angles of another triangle, what relation must these triangles bear to each other?

A. They must be equal.

Q. Supposing in this diagram the side *a b* equal to AB; the angle at *a* equal to the angle at A, and the angle at *b* equal to the angle at B; how can you prove that the triangle *a b c* is equal to the triangle ABC?



A. By applying *a b* upon its equal AB, the side *a c* will fall upon AC, and *b c* upon BC, because the angles *a* and A, *b* and B, are respectively equal; and as the sides *a c*, *b c*, take the same direction as the sides AC, BC, they must also meet in the same point in which the sides AC, BC, meet; that is, the point *c* will fall up-

on C; and the triangles will coincide throughout.

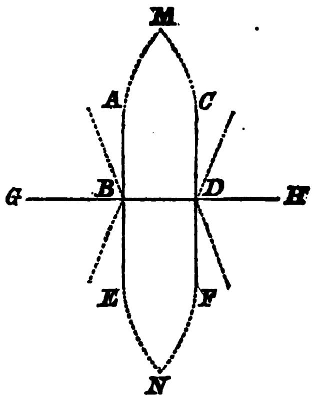
Q. What relation do you here discover between the equal sides and angles?

A. That the equal angles *c*, *C*, are opposite to the equal sides *a b*, AB respectively.

QUERY VII.

If two straight lines are both perpendicular to another straight line, what relation must they bear to each other?

A. They must be parallel.



Q. Let us suppose the two lines AB, CD, to be both perpendicular to a third line, GH; how can you convince me that AB and CD are parallel?

A. Because, if you extend AB and CD in the directions BE, DF, making BE and DF equal to BA and DC respectively, everything will be equal on both sides of the line GH. Now if the lines AB, CD, are not parallel, they must either be converging or diverging. If they are converging, AB and CD will, when sufficiently extended, cut each other somewhere in M;

but then the same must take place with the lines BE, DF, on the other side of the line GH, which will cut each other somewhere in N; and there would be two straight lines cutting each other in *two* points, which is impossible. If the lines AB, CD, were diverging, BE, DF, would be the same; but it is equally absurd to suppose two straight lines to diverge in *two* directions: consequently, the two straight lines, AB, DC, can neither be converging nor diverging, and therefore they must be parallel.

Q. Can two straight lines which cut each other, be perpendicular to the same straight line?

A: No.

Q. Why not?

A. Because, if they are both perpendicular to a third line, I have just proved that they must be parallel; and if they are parallel, they cannot cut each other.

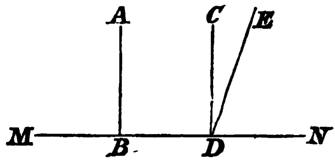
Q. From a point without a straight line, how many perpendiculars can there be drawn to that same straight line?

A. Only *one*.

Q. Why can there not more be drawn?

A. Because I have proved that two perpendiculars to the same straight line must be parallel to each other; and two lines, parallel to each other, cannot be drawn from one and the same point.

QUERY VIII.



If one of two parallel lines, for instance the line AB, is perpendicular to a third line MN, does it follow that the other parallel line CD, must also be perpendicular to it?

A. Yes. Because I have proved (query 7) that two lines, which are both perpendicular to a third line, must be parallel; and this is only the reverse of it.

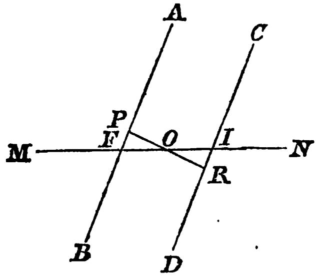
Q. Your answer is not quite satisfactory. For some might imagine it possible that *another* line, DE, likewise parallel to AB, could, at the same time, be perpendicular to MN.

A. But then there would be two straight lines, drawn from the same point D, parallel to the same straight line AB, which cannot be. (Truth 10.)

Will you now repeat your whole demonstration, and show that if one of two parallel lines is perpendicular to a straight line, the other line must also be perpendicular to it?

*　*　*

QUERY IX.



If a straight line MN, cuts two other straight lines at equal angles; that is, so as to make the angles CIN and AFN equal; what relation exists between these two lines?

A. They are parallel to each other.

Q. How can you prove it by this diagram? The line IF is bisected in *O*, and, from that point O, a perpendicular OP is dropped to the line AB, and afterwards extended until, in the point R, it strikes the line CD.

A. I should first observe that the triangles OPF and ORI are equal; because the triangle OPF has a side and two adjacent angles equal to a side and two adjacent angles of the triangle ORI, each to each. (Query 6.)

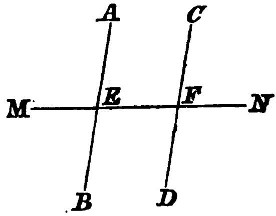
Q. Which is that side, and which are the two adjacent angles?

A. The side OI, which is equal to OF; because the point O bisects the line IF. One of the two adjacent angles is the angle IOR, which is equal to the angle FOP; because these angles

are opposite at the vertex; and the other is the angle OIR, which is equal to the angle OFP; because the angles CIN and AFN are, in the query, *supposed* to be equal; and CIN and OIR are opposite angles at the vertex.

Q. But of what use is your proving that the triangle ORI is equal to the triangle OPF?

A. It shows that since the triangle OPF is right-angular in P, the triangle OIR must be right-angular in R; for, in equal triangles, the equal angles are opposite to the equal sides (query 6); consequently the two lines AB, CD, are both perpendicular to the same straight line PR, and therefore parallel to each other.



Q. Supposing now two straight lines AB, CD, to be cut by a third line MN, so as to make the alternate angles AEF and EFD, or the angles BEF and EFC equal, what relation would the lines AB, CD, then bear to each other?

A. They would still be parallel.

Q. How do you prove this?

A. For if the angle AEF is equal to the angle EFD, the angles AEF and CFN will also be equal; because EFD and CFN are opposite angles at the vertex; and, in the same manner, if the angles BEF and EFC are equal, MEA and

EFC will also be equal; because MEA and BEF are opposite angles at the vertex. Therefore, in both cases, there will be two straight lines cut by a third line at equal angles, and consequently they will be parallel to each other.

Q. But there is one more case, and that is: *If the two straight lines AB, CD* (in our last figure), *are cut by a third line MN, so as to make the sum of the two interior angles AEF and EFC equal to two right angles, how will the straight lines AB, CD, then be situated with regard to each other?*

A. They will still be parallel to each other. For the sum of the two adjacent angles EFC and CFN is also equal to two right angles; and therefore, by taking from each of the equal sums the common angle EFC, the two remaining angles AEF and CFN must be equal (truth 7); and you have again the first case, viz: two straight lines cut by a third line at equal angles.

Q. Will you now state the different cases in which two straight lines must be parallel?

A. 1*stly, When they are cut by a third line at equal angles.*

2*dly, When they are cut by a third line so as to make the alternate angles equal; and*

3*dly, When the sum of two interior angles, made by the intersection of a third line, are together equal to two right angles.*

Q. Will you now repeat the demonstration of each separate case?

* * *

QUERY X.

Supposing the two straight lines CD, EF, to be cut by a third line AM *at unequal angles* ABC, BHE, *(Fig. I. and II.), or so as to have the alternate angles CBH and BHF, or DBH and BHE unequal; or in such a manner, that the sum of the two interior angles CBH and BHE (Fig. I.), or DBH and BHE (Fig. II.), is less than two right angles, what will then be the case with the two straight lines CD, EF?*

Fig. I.

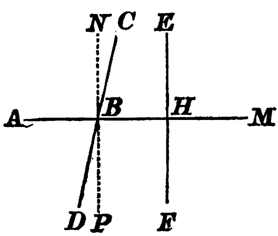
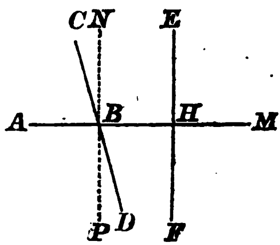


Fig. II.



A. They will, in every one of these cases, cut each other, if sufficiently extended.

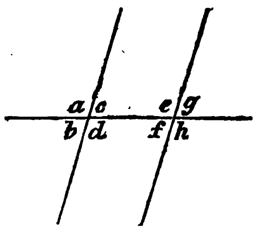
Q. How can you prove this?

A. By drawing, through the point B, another line NP at equal angles with EF, and which will then also make the alternate angles NBH and

BHF, or PBH and BHE equal; and the sum of the two interior angles NBH and BHE equal to two right angles: the line NP will be parallel to the line EF; and then, if the line CD would not cut EF, it would also be parallel to it, and there would be, through the same point B, two lines NP, CD, drawn parallel to the same straight line EF, which cannot be (truth 10): Therefore the line CD is not parallel to EF; and therefore it must cut the line EF.

QUERY XI.

Can you now tell the relation which the eight angles, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, formed by the intersection of two parallel lines, by a third line, bear to each other?



A. Yes. *In the first place, the angle* a *is equal to the angle* e; *the angle* c *equal to the angle* g; *the angle* b *equal to the angle* f; *and the angle* d *equal to the angle* h;—*2dly, the angles* a, d, e, h, *as well as the angles* b, c, f, g, *are respectively equal to one another; and finally, the sum of either* c *and* e, *or* d *and* f, *must make two right angles.* For if either of these cases were not true the lines would not be parallel. (Query 10.)

QUERY XII.

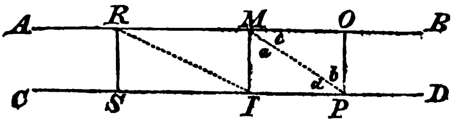
From what you have learnt of the properties of parallel lines, what law can you discover respecting the distance *they keep from each other?*

A. That parallel lines remain throughout equidistant.

Q. When do you call two lines equidistant?

A. When all the perpendiculars, dropped from one line to the other, are equal.

Q. How can you prove, that the perpendicular lines OP, MI, RS &c. are all equal?



A. By joining MP the two triangles MPO, MPI will have the side MP common; and the angle *a* is equal to the angle *b*; because *a* and *b* are alternate angles, formed by the two parallel lines MI, OP (query 11); and the angle *c* is equal to the angle *d;* because these angles are formed in a similar manner by the parallel lines AB, CD: therefore we have a side and the two adjacent angles in the triangle MPO, equal to a side and the two adjacent angles in the triangle MPI; consequently these two triangles must be equal; and the side OP opposite to the angle c, in the triangle MPO must be equal to the

side MI, opposite to the equal angle *d*, in the triangle MPI. In precisely the same manner I can prove that RS is equal to MI, and consequently equal to OP; and so I might go on, and shew that every perpendicular, dropped from the line AB to the parallel line CD, is equal to RS, MI, OP, &c. The two parallel lines AB and CD will therefore, throughout, be at an equal distance from each other; and the same can be proved of other parallel lines.

QUERY XIII.

If two lines are parallel to a third line, what relation must they bear to each other?

Fig. I.

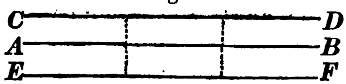
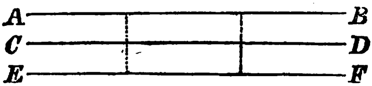


Fig. II.



A. They must be parallel to each other.

Q. How can you prove this?

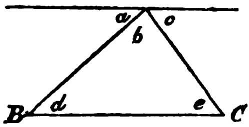
A. From the line CD being parallel to AB, it follows that every point in the line CD must be at an equal distance from the line AB; and because EF is also parallel to AB, every point in the line EF must also be at an equal distance from the line AB; and therefore (in *Fig. I.*) the *whole* distance between the lines CD and

EF, or (in *Fig. II.*) the *difference* between the equal distances, must be equal; that is, the lines CD, EF, must likewise be equidistant; and consequently, parallel to each other.

QUERY XIV.

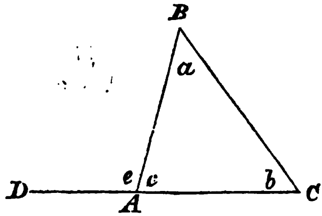
What is the sum of all the angles in every triangle equal to?

A. To two right angles.



Q. How do you prove this?

A. By drawing, through the vertex of the angle *b*, a straight line parallel to the basis BC, the angle *a* will be equal to the angle *d*, and the angle *c* to the angle *e* (query 11); and as the sum of the three angles *a*, *b*, *c*, is equal to two right angles (query 4), the sum of the three angles *d*, *b*, *e*, in the triangle, must also be equal to two right angles.*



Q. Can you now find out the relation, which the exterior angle *e*, bears to the two interior angles *a* and *b*?

A. The exterior

* The teacher may give his pupils an ocular demonstration of this truth, by cutting the three angles *b*, *d*, *e*, from a triangle, and then placing them along side of each other they will be in a straight line.

angle e, *will be equal to the sum of the two interior angles,* a *and* b.

Q. How do you prove this?

A. Because, by adding the angle *c* to the two angles *a* and *b*, it will make with them two right angles; and by adding it to the angle *e* alone, the sum of the two angles, *e* and *c*, will also be equal to two right angles, which could not be, if the angle *e* were not equal to the two angles *a* and *b* together. (Truth 2.)

Q. What other truths can you derive from the two which you have just now advanced?

A. 1. *That the exterior angle* e *is greater than either of the interior opposite ones,* a *or* b.

2. *If two angles of a triangle are known, the third angle is also determined.*

3. *When two angles of a triangle are equal to two angles of another triangle, the third angle in the one, will be equal to the third angle in the other.*

4. *No triangle can contain more than one right angle.*

5. *No triangle can contain more than one obtuse angle.*

6. *In a right-angular triangle, the right angle is equal to the sum of the two other angles.*

Q. How can you convince me of the truth of each of these assertions?

* * *

RECAPITULATION OF THE TRUTHS CONTAINED IN THE FIRST SECTION.

Can you now repeat the different principles of straight lines and angles which you have learned in this section.

Ans. 1. Two straight lines can cut each other only in *one* point.

2. Two straight lines, which have two points common, must coincide with each other throughout.

3. The sum of the two adjacent angles, which one straight line makes with another, is equal to two right angles.

4. The sum of all the angles, made by any number of straight lines meeting in the same point, and on the same side of a straight line, is equal to two right angles.

5. Opposite angles at the vertex are equal.

6. The sum of all the angles, made by the meeting of ever so many straight lines around the same point, is equal to four right angles.

7. When a triangle has one side and the two adjacent angles, equal to one side and the two adjacent angles in another triangle, the two triangles must be equal.

8. In equal triangles the equal angles are opposite to the equal sides.

9. If two straight lines are perpendicular to a third line, they must be parallel.

10. If one of two parallel lines is perpendicular to a third line, the other line must also be perpendicular to it.

11. If two lines are cut by a third line at equal angles, or so as to make the alternate angles equal, or so that the sum of the two interior angles, formed by the intersection of a third line, is equal to two right angles, the two lines must be parallel.

12. If two parallel lines are cut by a third line, the alternate angles are equal.

13. Parallel lines are throughout equidistant.

14. If two lines are parallel to a third line, they must be parallel to one another.

15. The sum of the three angles in a triangle is equal to two right angles.

16. If one of the sides of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.

17. The exterior angle is greater than any one of the interior opposite angles.

18. If two angles of a triangle are given, the third is determined.

19. There can be but *one* right angle, or *one* obtuse angle, but never a right angle *and* obtuse angle, in the same triangle.

20. In a right-angular triangle, the right angle is equal to the sum of the two other angles.*

* The teacher may now ask his pupils to repeat the demonstrations of these principles.

SECTION II.

OF EQUALITY AND SIMILARITY OF TRIANGLES.

PART I.

OF THE EQUALITY OF TRIANGLES.

Preliminary Remark. There are three kinds of equality to be considered in triangles, viz: equality of area without reference to the shape; equality of shape without reference to the area, or *similarity;* and equality of both shape and area, or *coincidence.* All questions asked in this section will refer only to the last two kinds of equality; and those in the first part, only to the *coincidence* of triangles.

QUERY I.

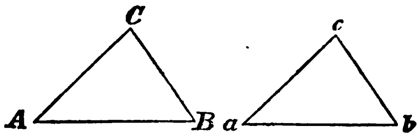
If in two triangles, two sides of the one are equal to two sides of the other, each to each, and the angles which are included by them also equal to one another, what relation will these two triangles bear to each other?

Ans. They will be equal to each other in all their parts, that is, they will coincide with each other throughout.

Show me that this must be the case with the two triangles, ABC and *a b c*, in which we

4*

will suppose the side AB = $a\,b$, AC = ac, and the angle at A equal to the angle at a.



A. By placing the line $a\,c$ upon its equal, AC, the angle at a will coincide with the angle at A, because these two angles are equal; and the line ab will fall upon the line AB; and as $a\,b$ = AB the point b will fall upon B; that is, the three points of the triangle $a\,b\,c$ will fall upon the three points of the triangle ABC, thus:

The point a upon A,
" b " B,
" c " C;

consequently these two triangles will coincide.

Q. What remark can you make with respect to the sides and angles of triangles which coincide with each other?

A. That the equal sides, $c\,b$, CB, are opposite to the equal angles, at a and A.

QUERY II.

If one side and the two adjacent angles in one triangle, are equal to one side and the two adjacent angles in another triangle, each to each, what relation will the two triangles bear to each other?

A. They will be equal, and the angles opposite to

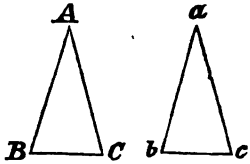
the equal sides will also be equal, as has been proved in the 1st Section. (Query 6.)

QUERY III.

What remark can you make with respect to the two angles at the basis of an isosceles triangle?

A. They are equal to each other.

Q. How can you prove this?



A. Suppose we had two equal isosceles triangles, ABC and *a b c*, or as it were, another impression, *a b c*, of the triangle ABC, that is,

The side *a b* = AB,
" *a c* = AC,
" *b c* = BC.

The angle at *a* = angle at A,
" *b* = " B,
" *c* = " C.

Then the sides, AB, AC, *a b*, *a c*, being all equal to one another, and the angle at *a* remaining the same, whichever way we place it, the whole of the two triangles, *a b c* and ABC, will still coincide, when *a b c* is placed upon ABC in such a manner that *a c* will fall upon AB, and *a b* upon AC (for you will still have two sides and the angle which is included by them in the one, equal to two sides and the angle which is included by them in the other); and therefore the angle at *c*

must coincide with the angle at B; and as the angle at c is only as it were, another impression of the angle at C, B and C must also coincide; that is, the two angles, B and C, at the basis of the isosceles triangle ABC, must be equal: and the same can be proved of the two angles at the basis of every other isosceles triangle.

QUERY IV.

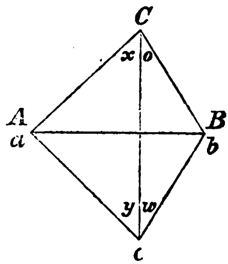
If the three sides of one triangle are equal to the three sides of another, each to each, what relation will the two triangles bear to each other?

A. They will coincide throughout; that is, their angles will also be equal, each to each.

Q. How can you prove this, for instance, of the two triangles ABC and abc, in which we will suppose the side $AB = a\,b$,

$AC = a\,c$,

$BC = b\,c$?



That you may easier find out your demonstration, I have placed the two triangles, as you see, along side of each other, with their bases, AB and $a\,b$, together, and have joined their opposite vertices, C and c, by the straight line Cc. What can you now observe with regard to the two triangles ACc and BCc?

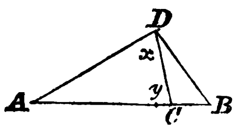
A. Both are isosceles; for the sides, AC and *a c*, BC and *b c*, are respectively equal; and, therefore, the angles *x* and *y*, and *o* and *w*, must be equal, each to each; and as the angle *x* is equal to the angle *y*, and the angle *o* equal to the angle, *w*, the sum of the two angles *x* and *o*, that is, the *whole* angle ACB must be equal to the sum of the two angles *y* and *w*, that is, to the *whole* angle *a c b*; and the two triangles, ABC and *a b c*, having two sides and the angle which is included by them in the one, equal to the two sides and the angle which is included by them in the other, each to each, must coincide throughout. (Query 1, Sect. II.)

QUERY V.

Which of two angles in a triangle is the greater, that which is opposite to the smaller, or that which is opposite to the greater side?

A. That which is opposite to the greater side.

Q. How can you prove it?



A. Because, if in any triangle, for instance the triangle ABD, one side AB, is greater than another AD, the side AB will contain a part which is equal to AD,* and therefore, by taking upon AB the distance AC equal to AD, and join-

* If the magnitude A, is greater than B, A must contain a *part*, equal to B.

ing DC, the triangle ACD will be isosceles, and the angle x will be equal to the angle y (Query 3, Sect. II.) ; and as the exterior angle y must be greater than the interior opposite angle at B, in the triangle DBC, (Query 14, Sect. I.) the angle at x will also be greater than the angle at B ; and the angle ADB being greater still, than the angle x, must consequently be still more so than the angle at B ; that is, the angle ADB, opposité to the greater side AB, will be greater than the angle at B, opposite to the smaller side AD : and the same can be proved of two *unequal* sides in any other triangle.

Q. What truth can you directly derive from this, respecting the three angles and sides of a triangle ?

A. That the greatest of the three angles must be opposite to the greatest of the three sides. For if the side AD, for instance, is greater than the side DB, I have proved that the angle at B, opposite to the side AD, must also be greater than the angle at A, opposite to the side DB ; and as the side AB is greater still than AD, the angle ADB, opposite to AB, will be greater still than the angle at B, and will therefore be the greatest angle in the triangle ABD.

Q. But does it follow, that because the greater angle is opposite to the greater side, the greater side is opposite to the greater angle ?

A. *Yes.* For if, in the last figure, the angle at B is greater than the angle at A, it is evident that the side AD, opposite to the greater angle at B, must be greater than the side BD, opposite to the smaller angle at A. For the side BD can neither be *equal* to AD, (because in this case the triangle ABD would be isosceles and the angles at B and A equal) nor can it be *greater* than AD; for then I have proved, that the angle at B would be *smaller* than the angle at A; therefore the side BD can neither be *equal* to, nor *greater* than AD; therefore it must be *smaller* than AD; and the greater side AD must be opposite to the greater angle at B.

Q. And is it also true that the *greatest* side of a triangle is opposite to the *greatest* angle in it?

A. Yes. For if the angle at B is greater than the angle at A, the side AD, opposite to the angle at B, will also be greater than the side DB, opposite to the angle at A; and if the angle ADB is greater still than the angle at B, the side AB, opposite to this angle, will be greater still than the side AD; that is, the greatest side will be opposite to the greatest angle.

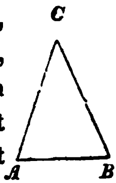
Q. From what you have learned of the relation which exists between the sides and angles of a triangle, can you now tell which of the sides of a right-angular triangle will be the greatest?

A. Yes. That which is opposite to the right angle.

Q. Why?

A. Because in a right-angular triangle the right angle is greater than either of the two other angles. (Query 14, conseq. 6, Sect. I.)

QUERY VI.

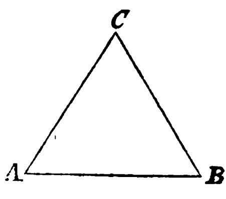


It has been proved before (Query 3, Sect. II.), that in an isosceles triangle, the angles at the basis are equal; can you now prove the reverse; that is, that a triangle must be isosceles when it contains two equal angles?

A. Yes. Because if either of the two sides, AC, BC, were greater than the other, the angle opposite to that side, would also be greater than the angle which is opposite to the other side.

Q. *If the three angles in a triangle are equal to one another, what relation will the* sides *bear to each other?*

A. *They will also be equal, and the triangle will be equilateral.*



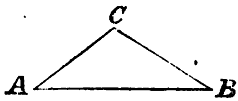
Q. How can you prove this?

A. If in the triangle ABC, for instance, the angle at A is equal to the angle at B, I have just proved that the side BC must also be equal to the side AC; and if the angle at B is also equal to the angle at C, the side AC must likewise be equal to the side AB: that is, the three sides AB, BC, AC, will be equal, and the triangle will be equilateral.

QUERY VII.

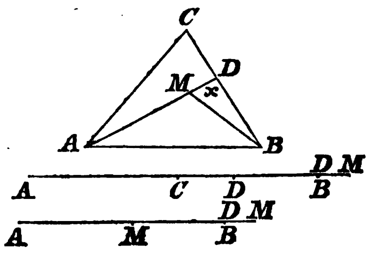
Can any one side of a triangle be greater than, or equal to, the sum of the two other sides.

A. No. A straight line being the shortest way from one point to another, it follows, that in any triangle, ABC for instance, the side AB is smaller than the sides AC and BC together.



QUERY VIII.

If from a point M, in a triangle ABC, two lines, AM, BM, be drawn to the two extremities of any side, AB, in that triangle, what relation will the angle AMB, made by these two lines, bear to the angle ACB, which is opposite to the side AB in the triangle? and what do you observe with regard to the sum of the two lines, AM and MB, which include the angle AMB, and that of the two sides, AC, BC, of the triangle which include the angle ACB?



A. The angle AMB made by the lines AM, MB, will always be greater than the angle ACB, opposite to the side AB, in the triangle ABC; but the sum of the two lines AM, MB, will in all cases be smaller than the sum of the two sides AC, CB, of the triangle.

Q. How can you prove both your assertions?

A. The exterior angle x is greater than the interior opposite angle ACD, in the triangle ACD (Query 14, Sect. I.); and for the same reason is the exterior angle AMB greater than the interior opposite angle x, in the triangle MDB; and therefore the angle AMB is greater still, than the angle ACB in the triangle ABC. 2dly, The two sides, AC and CD, of the triangle ACD must, together, be greater than the third side AD, or its two parts AM and MD; and for the same reason, the two sides, MD and DB, in the triangle MDB, must, together, be greater than the third side MB, which may be written thus:

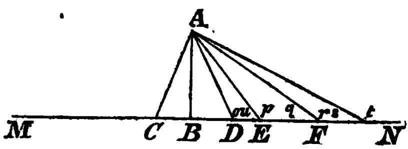
$$AC + CD > AM + MD,$$
$$BD + DM > MB.$$

(See Notations and Significations.) Now, it is evident that the sum obtained by adding the greater to the greater must be greater, than the sum obtained by adding the smaller to the smaller; that is, a line which is as long as the four lines AC, CD, BD, DM, together, must be greater than one only as long as the lines AM, MD, MB, together. (See the figure.) Now, by cutting off from each of the two unequal lines

the same piece DM, the remainder of the longer line must be greater than the remainder of the shorter line; that is, AC + CD + DB, or, (which is the same) AC + CB, (because CB is equal to CD + DB), must be greater than AM + MB; or, in other words, a line, which is equal to the sum of the two sides AC, CB, of the triangle ACB, must be longer than one, which is equal to the sum of the two lines AM and MB.

QUERY IX.

If from a point A, without a straight line MN, you drop a perpendicular, AB, to that line; and, at the same time, draw other lines, AD, AE, AF, &c., obliquely to different points, D, E, F, &c., in the same straight line; which will be the shortest, the perpendicular, or one of the oblique lines?



A. The perpendicular will be the shortest.

Q. How can you prove it?

A. Because the triangles, ABD, ABE, ABF, ABN, &c., are all right-angular in B; and in every right-angular triangle the greatest side is opposite to the right angle. (Page 47, Ques. 2d.)

Q. And what other truths do you derive from the one you have just mentioned?

A. 1*st. That the perpendicular AB measures the distance of the point A from the line MN;* for it is the *shortest* line, that can be drawn from that point to that line.

2*dly, The angles* o, p, r, t, *&c., are all obtuse,* because they are exterior angles of the right-angular triangles, ABD, ABE, ABF, &c., and, therefore, greater than the interior opposite right angle at B.

3*dly, That the angles* o, p, r, t, *&c., become successively greater, and the angles* u, q, s, &c., *smaller, as the lines AD, AE, AF, &c., are drawn farther from the perpendicular.* For the exterior angle *p*, is greater than the interior opposite one *o*, in the triangle ADE; the exterior angle *r* is greater than the interior opposite one *p*, in the triangle AEF; the exterior angle *t*, again, is greater than the interior opposite one *r*, in the triangle AEN, and so on.

4*thly, The oblique lines, AD, AE, AF, &c., become successively greater, as they are drawn farther from the perpendicular;* that is, the line AD is greater than the line AB, the line AE than the line AD, the line AF than the line AE, and so on. For the angles *o*, *p*, *r*, &c., are all obtuse, and become successively greater, as the triangles ADE, AEF, &c., are more remote from the perpendicular; and, therefore,

the sides AE, AF, AN, opposite to these angles, will become greater with them.

5thly, *The straight lines, AC, AD, drawn on both sides of, and at an equal distance from, the perpendicular AB, must be equal.* For the two triangles ABC, ABD, have the side AB common, and the side BC equal to the side BD (because the lines, AC, AD, are at an equal distance from the perpendicular AB), and because the line AB is perpendicular to CD, the angle ABC, included by the sides AB, BC, in the triangle ABC, is equal to the angle ABD included by the sides AB, BD, in the triangle ABD; consequently, these two triangles must be equal; and, the third side AC, in the one triangle, must be equal to the third side AD in the other. (Query 1, Sect. II.)

6thly, *There is but one point in the line MN, on each side of the perpendicular, such, that a straight line drawn from it to the point A, is of a given length.* This follows from No. 4.

7thly, *There is but one point in the line MN, on each side of the perpendicular, in which a line drawn to the point A, forms with the line MN an angle of a given magnitude.* This follows from No. 3.

QUERY X.

If two sides, and the angle which is opposite to the greater of them, in one triangle, are equal

to two sides and the angle which is opposite to the greater of them in another, each to each, what relation will these two triangles bear to each other?

A. They will coincide with each other in all their parts; that is, they will be equal to each other.

Q. How can you prove it?

A. Because, if in a triangle ABC, for instance, you have the sides AB and AC, and the angle at B, which is opposite to the greater side AC, given, the whole triangle will be *determined.*

Fig. I.

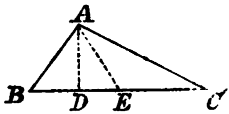
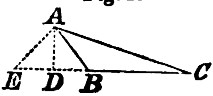


Fig. II.

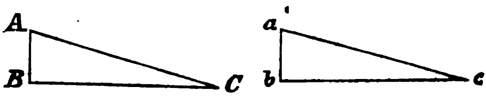


For, in the first place, by the angle at B, the direction of the sides AB, BC, is determined. *2dly,* By the length of the side AB, the distance of the point A from the line BC, is determined: Now, if you imagine the perpendicular AD to be dropped upon BC (Fig. I.), or, if the angle at B be obtuse (as in Fig. II.) on its further extension BE, there can be but *one* point in the line BC, on this side of the perpendicular, from which a line drawn to the point A, is as long as the line AC (see consequence 6th of the preceding Query); and therefore, 3*dly*, by the length of the line AC, the point C, and thereby the whole of the third line BC, will also be determined.

Q. But is it not possible for this line AC, to fall on the other side of the perpendicular?

A. No. Because the line AC, being greater than the line AB, would in this case be farther from the perpendicular, than the line AB (conseq. 4, preceding Query), and the angle at B would then fall *without* the triangle: and because the whole triangle ABC is entirely determined, when two of its sides, and the angle, which is opposite to the greater of them, are given: therefore, all triangles, in which these three things are equal, must be equal to one another.

Q. What truth can you infer from this, respecting the case where the hypothenuse,* and one side of a right-angular triangle, are equal to the hypothenuse and one of the sides in another right-angular triangle?



A. That these two right-angular triangles must be equal to each other. For, in this case, we have two sides, and the *right* angle which is opposite to the *greater* of them in the one, equal to two sides, and the angle which is opposite to the greater of them, in the other.

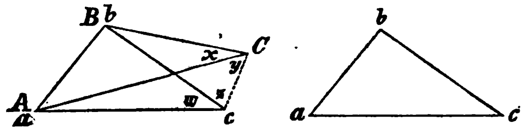
* In a right-angular triangle, the side, which is opposite to the right angle, is called *hypothenuse*.

Q. But if in Fig. II. (page 54) the two sides AC, AB, and the angle at C, opposite to the *smaller* side AB, be given, would not this be sufficient to determine the triangle ABC?

A. No. For the two lines, AB, AE, being equal, there would be two triangles, ABC and AEC, possible, containing the same three things, and it would be doubtful which of the two triangles was meant.

QUERY XI.

If you have two sides, ab, bc, *of a triangle* abc, *equal to two sides AB, BC, of another triangle ABC, each to each; but the angle ABC included by the two sides, AB, BC, in the triangle ABC, greater than the angle* abc, *included by the sides,* ab, bc, *in the triangle* a b c, *what remark can you make with regard to the two sides,* a c, *AC, which are respectively opposite to those angles?*



A. That the side a c, *opposite to the smaller angle* a b c, *in the triangle* a b c, *is smaller than the side AC, opposite to the greater angle ABC, in the triangle ABC.*

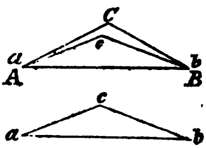
Q. How do you prove this?

A. By placing the triangle *a b c* upon the triangle ABC, with the side *a b* upon AB (its equal), the side *b c* will fall *within* the angle

ABC, because the angle *a b c* is smaller than the angle ABC; and the extremity *c*, of the line *b c*, will either fall without the triangle ABC, as you see in the figure before you, or within it, or it may also fall upon the line AC itself.

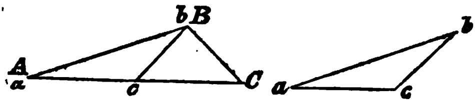
1*st*. If it falls without the triangle ABC, by imagining the line C*c* drawn, the triangle *c* BC will be isosceles; for we have supposed the sides, *b c*, BC, to be equal; therefore, because the angles at the basis of an isosceles triangle are equal (Query 3, Sect. II.), the angle *z* is equal to the *sum* of the two angles *x* and *y*; consequently greater than the angle *y alone*; and if the angle *z* is greater than the angle *y*, the two angles *z* and *w* together, will be greater still than the same angle *y*; therefore, in the triangle AC *c*, the angle A *c* C is greater than the angle AC *c*; consequently the side AC, opposite to the greater angle A *c* C, must be greater than the side *a c*, opposite to the smaller angle AC *c*.

2*dly*, If the extremity of the line *b c* falls within the triangle ABC, the sum of the two sides, *a c*, *b c*, must be smaller than the sum of the two sides, AC, BC (Query 8, Sect. II.); therefore, by taking from each of these sums the equal lines, *b c*, BC, respectively, the remainder AC, of the greater sum (AC + BC),

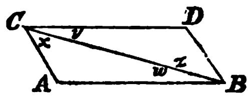


will be greater than the remainder $a\,c$, of the smaller sum $(a\,c + b\,c)$.

Finally, If the point c falls upon the line AC itself, it is evident that the whole line AC must be greater than its part A c.



QUERY XII.



If in a parallelogram ACDB, you draw a diagonal CB, what relation will the two triangles, ABC, CDB, bear to each other?

A. They will be equal to one another, and the parallelogram will be divided into two equal parts.

Q. How can you prove this?

A. The two triangles, ABC and CDB, have the side CB common; and the angle y is equal to the angle w; because y and w, are alternate angles, formed by the intersection of the two parallel lines, CD, AB, by a third line CB; and the angle x is equal to the angle z, because these two angles are formed in a similar manner, by the parallel lines AC, DB (Query 11, Sect. I.): therefore, because the triangle ABC has a side CB, and the two adjacent angles, x and

w, equal to the side CB, and the two adjacent angles, *y* and *z*, in the triangle CDB, each to each, these two triangles must be equal (Query 6, Sect. I.), and the diagonal CB, must divide the parallelogram into two equal parts.

Q. *What other properties of a parallelogram can you infer from the one just learned?*

1*st*, *The opposite sides of a parallelogram are equal: that is, the side CD is equal to the side AB, and the side CA to the side DB;* for in the equal triangles, ABC, CDB, the equal sides must be opposite to the equal angles. (Conseq. of Query 1, Sect. II.)

2*dly*, *The opposite angles in a parallelogram are equal;* for in the two equal triangles, ABC, CDB, the same side CB, is opposite to each of the angles, at D and A. (Conseq. of Query 6, Sect. I.)

3*dly*, *By one angle of a parallelogram, all four are determined;* for the sum of the four angles in a parallelogram is equal to *four right angles;* because the sum of the three angles in each of the two triangles, ABC, CDB, is equal to *two* right angles. Now, if the angle at D, for instance, is known, the angle at A is equal to it; and there remain but the two angles, ACD and ABD, each of which must be equal to half of what is wanting to complete the sum of the four right angles.

Q. *If you have a quadrilateral, in which the*

opposite sides are respectively equal, does it follow that the figure must be a parallelogram?

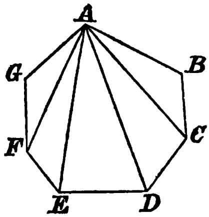
A. Yes. For if, in the last figure, you have the side CD equal to the side AB, and the side AC equal to the side BD; by drawing the diagonal BC, you have the triangle ABC with all three sides respectively equal to the three sides of the triangle CDB; therefore, these two triangles must be equal; and the angle y, opposite to the side DB, must be equal to the angle w, opposite to the equal side AC; and the angle x opposite to the side AB, equal to the angle z, opposite to the equal side CD; that is, the alternate angles, y and w, x and z, are respectively equal: therefore, the side CD is parallel to the side AB, and the side AC to the side BD (See page 31, 2dly); and the figure is a parallelogram.

Q. If in a quadrilateral there are but two sides equal and parallel, what will then be the name of the figure?

A. It will still be a parallelogram. For if, in the last figure, the side CD is equal and parallel to AB, by drawing the diagonal CB, you have the two sides, CB and CD, in the triangle CDB, equal to the two sides, CB, AB, in the triangle ABC, each to each; and because the side CD is parallel to the side AB, the included angle y is equal to the included angle w; therefore, the two triangles must be equal (Query 1, Sect. II.), and the side AC is

also equal and parallel to the side DB, as before.

QUERY XIII.



If from one of the vertices of a rectilinear figure, diagonals are drawn to all the other vertices, into how many triangles will this rectilinear figure be divided?

A. Into as many, as the figure has sides less two. For it is evident, that if from the vertex A, for instance, you draw the diagonal AF, AE, AD, AC, to the vertices F, E, D, C, each of the two triangles AGF, ABC, will need for its formation two sides of the figure, and a diagonal; but then every remaining side of the figure will, together with *two* diagonals, form a triangle; therefore there will be as many triangles formed, as there are sides less the two, which are additionally employed in the formation of the two triangles AGF, ABC.

Q. And what is the sum of all the angles, BAG, AGF, GFE, FED, EDC, DCB, CBA, equal to?

A. To as many times two right angles as the figure ABCDEFG has sides less two. For as every rectilinear figure can be divided into as many triangles, as there are sides less two; and because the sum of the three angles in each tri-

angle is equal to two right angles (Query 14, Sect. I.), there will be as many times *two right angles* in all the angles of your figure, as there are triangles; that is, twice as many right angles, as the figure has sides less two.

SECTION II.

PART II.

OF GEOMETRICAL PROPORTIONS,* AND SIMILARITY OF TRIANGLES.

WHENEVER we compare two things with regard to their magnitude, and inquire *how many times* one is greater than the other, we determine the ratio, which these two things bear to each other. If in this way, we find out that the one is twice, three, four, &c., times greater than the other, we say that these things are in the ratio of one to two, to three, to four, &c. *e. g.* If you compare the fortunes of two persons, one

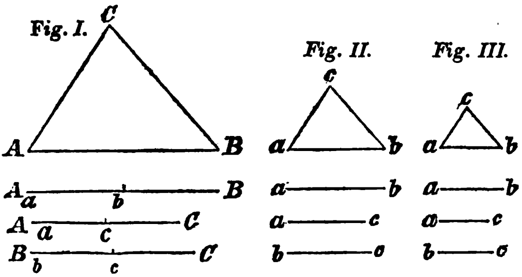
* It is the design of the author to give here a perfectly *elementary* theory of geometrical proportions, and to establish every principle *geometrically*, and by simple *induction*. Intending the above theory for those who have not yet acquired the least knowledge of Algebra, he is not allowed to identify the theory of proportions with that of algebraic equations (as it is done by some writers on Mathematicks), and then to find out the principles of the former by an analysis of the latter. There are several disadvantages inseparable from the algebraic method of considering a ratio as a fraction, besides the difficulty of making such a theory accessible to beginners. Neither can an algebraic demonstration be made obvious to the eye like a geometrical one.

of whom is worth $10000, and the other $20000, you say, that their fortunes are in the ratio of one to two. Or if you compare two lines, one of which is two, and the other six feet long, you say of these lines, that they are in the ratio of one to three, because the second line is three times as long as the first.

It frequently occurs, that two things are to each other in the same ratio, in which two others are; we then say, that these things are in *proportion*. This is frequently the case in the fine arts; but particularly in the science of Geometry, from which these proportions are called *geometrical*. To give an example: If you draw a house, you must draw it *according to a certain scale;* that is, you will draw it one thousand, two thousand, three thousand, &c. times smaller than the building itself: but then you will be obliged to reduce every part of it *in the same proportion*. If, for instance, you draw the front of the house one thousand times smaller than the original, you must reduce the windows, doors, and every other part, *in the same ratio*. If, on the contrary, the windows were reduced *two* thousand times, whilst the doors and other parts were reduced only *one* thousand times, your picture would be *out of proportion*, because the different parts would be reduced by *different* ratios. In this case your

picture would be distorted; and would *not resemble the original.*

The same is the case with the resemblance, produced in any other kind of drawings; but particularly in geometrical figures.



If the two triangles, ABC, *a b c*, are to be similar to each other, it is necessary that they should be constructed after the same manner, and that the side AC should be exactly as many times greater than the side *a c*, as the side BC is greater than *b c*, and the side AB, than *a b*. If (Fig. I. and II.) the side AB, for instance, be twice as great as the side *a b;* that is, if the side *a b* be half of the side AB, the side *a c* must also be half of the side AC, and the side *b c* half of the side BC; that is, the three sides, *a b*, *a c*, *b c*, of the triangle *a b c*, must be in proportion to the three sides, AB, AC, BC, of the triangle ABC. Again, if (Fig. I. and III.) the side AB be three times as great as the side *a b;* that is, if the side *a b* be one third of the side

AB, the side *a c* must also be one third of the side AC, and the side *b c* one third of the side BC; or the triangles *a b c*, ABC, would not be similar to each other. The same holds true of all other geometrical figures, composed of any number of sides. If they are similar, their sides are proportional to each other.

There are different ways of denoting a geometrical proportion. Some mathematicians would express the proportionality of the sides, *a b*, *a c*, of the triangle *a b c* (Fig. II.), to the sides AB, AC, of the triangle ABC (Fig. I.), in the following manner:

$$AB : ab :: AC : ac;$$

or,

$$AB \div ab :: AC \div ac,$$

and also

$$AB : ab = AC : ac,\text{*}$$

which is read thus:

AB is to *a b*, as AC is to *a c*.

As a proportion is nothing else than the equality of two ratios, the third way of denoting a proportion, in which the sign of equality is put between the two ratios, seems to be the most natural. The reason why the sign of division (see Notation and Significations), is put

*The first manner of expressing a proportion is now in general use among the English and French mathematicians; the second is sometimes met with in old English writers, and the third way is adopted in Germany.

between the two terms, *AB*, *a b*, of a ratio, is obvious; for a ratio points out how many times one term (the side *a b*) is contained in another, (the side AB).

The first and fourth term of a proportion, together, are called *extremes*; because one of them stands at the *beginning*, and the other at the *end*, of a proportion: the second and third terms, standing in the *middle*, are, together, called the *means*.

The following principles of geometrical proportions ought to be well understood and remembered.

1st. It is important to observe, that in every geometrical proportion the two ratios may be inverted; that is, instead of saying,

$$AB : ab = AC : ac,$$

you may say, $ab : AB = ac : AC$.

For the order of terms being changed in *both* ratios, they will still be equal to one another; but leaving one ratio unaltered, if you change the order of terms in the other, the proportion will be destroyed. You cannot say,

$$ab : AB = AC : ac;$$

for the *smaller* side, *a b*, is contained *twice* in the greater side AB (Fig. I. and II.); but the *greater* side AC, is not contained *once* in the smaller side *a c*.

2d. Another remarkable property of geometrical proportions is, that you may change the order of the means, or extremes, without de-

stroying the proportion. Thus the proportion

$$AB : ab = AC : ac \quad \ldots\ldots\ldots \quad \text{(I.)}$$

may be changed into

$$AB : AC = ab : ac \quad \ldots\ldots\ldots \quad \text{(II.)}$$

or, by changing the extremes into

$$ac : ab = AC : AB \quad \ldots\ldots\ldots \quad \text{(III.)}$$

The reason, why you have a right to do this, is easily comprehended. If, in the first proportion, the side AB (Fig. I.), is exactly as many times greater than the side $a\,b$ (Fig. II.), as the side AC is greater than the side $a\,c$, the ratios of AB to AC will be the same as that of $a\,b$ to $a\,c$. In Fig. I. and II., for instance, we have $a\,b$ equal to one half of *AB;* consequently $a\,c$ is also equal to one half of *AC;* and therefore, let the ratio of the two lines, AB to AC, be whatever it may, their halves, $a\,b$ and $a\,c$, must be in the same ratio. No one will deny, that the same ratio, which *one* dollar bears to *one* hundred dollars, exists between *one half* dollar and *one hundred half* dollars; or that one hundred dollars are just as many times greater than one dollar, as one hundred half dollars are greater than one half dollar. The second proportion would still be correct, if, as in Fig. I. and III., the sides, AB, AC, were *three* times as great as the sides, $a\,b$, $a\,c$; for then the thirds of AB and AC would still be in the same proportion as the whole lines AB and AC. Nothing can now be easier than to extend this mode

of reasoning, and show the generality of the principle here advanced. The correctness of the third proportion might be proved precisely in the same manner as that of the second; for the third proportion differs from the second only *in the order* in which the two ratios are placed; and of two equal things, it does not matter which you put first. The correctness of the second proportion proves, therefore, that of the third proportion.

3d. If you have two geometrical proportions, which have one ratio common, the two remaining ratios will, again, make a proportion; for if two ratios be equal to the same ratio, they must be equal to each other. (See Axioms, Truth 1). If you have the two proportions:

$$AB : ab = AC : ac$$
$$AB : ab = BC : bc$$

you will also have the proportion

$$AC : ac = BC : bc.$$

For an *illustration* of this principle, we may take the two triangles, ABC, abc (Fig. I. and II.): If the sides, AB and ab, are in proportion to the sides, AC and ac, and also in proportion to the sides, BC and bc, the three sides of the triangle ABC, will be in proportion to the three sides of the triangle abc; therefore, any *two* sides of the first triangle will be in proportion to the two corresponding sides of the other triangle.

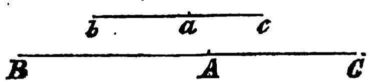
4th. Another important principle of geometrical proportions is this: If you have several geometrical proportions, of which the second has a ratio common with the first, the third a ratio common with the second, the fourth a ratio common with the third, and so on, the sum of all the first terms of these proportions will bear the same ratio to the sum of all the second terms, as the sum of all the third terms does to the sum of all the fourth terms; that is, the *sums* will again make a proportion.

To prove this, we will, in the first place, consider the simplest case; that of *two* proportions only; and the easier to comprehend it, take the same two proportions which we have just had under consideration, viz:

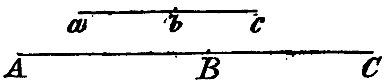
$$a\,c : \mathrm{AC} = a\,b : \mathrm{AB}$$
$$a\,b : \mathrm{AB} = b\,c : \mathrm{BC}.$$

We know, from the two triangles, ABC and $a\,b\,c$ (Fig. I. and II.), that, in the first proportion, $a\,c$ is half of AC; consequently $a\,b$ is also half of AB, and, in the second proportion, $b\,c$ is half of BC. Thus, each of the two first terms, $a\,b$, $a\,c$, is half of its second term; and consequently each of the third terms, $b\,c$, $a\,b$, is half of its corresponding fourth term; therefore, adding $a\,b$ and $a\,c$ together, their *sum* will be just one half of the sum of AB and AC; and so will $b\,c$ and $a\,b$, be, together, one half

of the sum of BC and AB. For the sake of illustration, you may measure off the length of



a b and *a c*, upon the line *b c*, and the length of AB and AC, on another line BC; and you will find that the line *b c* will be exactly one half of the line BC. For the line *b c*, composed of two *parts*, *a b*, *a c*, each measuring exactly one half of the corresponding two parts, AB, AC, of which the line BC is composed, must evidently be one half of the *whole* line BC. In the same way you may convince yourself that the



line *a c*, composed of the two parts, *a b* and *b c*, will measure one half of the second line AC, composed of the two parts, AB and BC: and therefore you will have

$$\overline{ab + ac} : \overline{AB + AC} = \overline{bc + ab} : \overline{BC + AB}.*$$

Although, in our example, we have chosen a proportion in which the first and third terms are exactly one half of the second and fourth terms, yet it is easy to perceive, that the same course

*The lines over $ab + ac$, AB + AC, &c., mark that $ab + ac$, AB + AC, &c., are but single lines composed of the two parts, ab, ac, and AB, AC, &c.

of reasoning will apply to any other two proportions. Thus, if the first terms in the above proportions were one third, or one fourth, or one fifth, &c., of the corresponding second terms, the *sum* of the first terms would also be one third, or one fourth, or one fifth, &c., of the sum of the second terms; and the same would be the case with regard to the sum of the third and fourth terms. It is also evident, that our principle would still hold true, if, instead of two proportions, we had three, four, or more proportions given, of which two and two had a common ratio. If, for instance, we had the *three* proportions:

$$\begin{aligned} ac : \mathrm{AC} \quad & ab : \mathrm{AB} \\ ab : \mathrm{AB} = & \; bc : \mathrm{BC} \\ bc : \mathrm{BC} = & \; ac : \mathrm{AC}, \end{aligned}$$

we should, according to our principle, have

$$\overline{bc + ab + ac} : \overline{\mathrm{BC} + \mathrm{AB} + \mathrm{AC}} = \overline{ac + bc + ab} : \overline{\mathrm{AC} + \mathrm{BC} + \mathrm{AB}}.$$

Each of the three lines, $b\,c$, $a\,b$, $a\,c$, would be one half of its corresponding part in the second term; and in the same way would each of the three lines, $a\,c$, $b\,c$, $a\,b$, be one half of its corresponding part in the fourth term; and therefore, the *sum* of the three lines, $b\,c$, $a\,b$, $a\,c$, or, which is the same, a line as great as the three lines, $b\,c$, $a\,b$, $a\,c$, together, would be one half of the sum of the three lines, BC, AB, AC, or a line as great as the three lines, BC, AB, AC,

together; and the same would be the case with the sum of the third and fourth terms. And in like manner could this principle be extended to four, five, six, and more proportions.

5th. Another principle, which it is important to recollect, is, that by adding the second term of a proportion once, or any number of times, to the first term, and the fourth term the same number of times to the third term, you will still have a proportion. To give an example: In the proportion

$$\mathrm{AB} : a\,b = \mathrm{AC} : a\,c,$$

let there be added the second term $a\,b$, in the first place, *once* to the first term AB; and the fourth term $a\,c$ also *once* to the third term AC. Our proportion will then be:

$$\overline{\mathrm{AB} + a\,b} : a\,b = \overline{\mathrm{AC} + a\,c} : a\,c,$$

in which the first term $\overline{\mathrm{AB}+a\,b}$ instead of being (as it was before, Fig. I. and II.) only twice as great as $a\,b$, will, by the addition of the term $a\,b$ itself, be three times as great as $a\,b$; and for the same reason will $\overline{\mathrm{AC}+a\,c}$, be three times as great as $a\,c$. The two new ratios

$$\overline{\mathrm{AB} + a\,b} : a\,b, \text{ and}$$

$$\overline{\mathrm{AC} + a\,c} : a\,c,$$

will therefore be equal, and consequently make a proportion. The same would be the case, if instead of adding the second and fourth terms

once, you would add them *twice* respectively to the first and third terms; with the only difference, that the first term, $\overline{AB+2ab}$, would then be four times as great as the second term ab. A similar change would take place with regard to the third term, $\overline{AC+2ac}$, which would then be four times as great as the term ac; and you would have the proposition:

$$\overline{AB+2ab} : ab = \overline{AC+2ac} : ac.$$

If the second term were added *three* times to the first term, the first term, $\overline{AB+3ab}$, would be five times as great as ab; and the third term, $\overline{AC+3ac}$, would also be five times as great as ac; and so on.—

In precisely the same manner you may prove that by adding the *first* term once, or any number of times, to the *second* term, and the *third* term the same number of times to the *fourth* term, the result will still be a proportion. Thus, our proportion

$$AB : ab = AC : ac,$$

may be changed into

$$AB : \overline{ab+AB} = AC : \overline{ac+AC},$$

or into $AB : \overline{ab+2\,AB} = AC : \overline{ac+2\,AC}$, &c.

It it also evident that the same principle will hold of any other geometrical proportion.*

* The teacher had better show this to the pupil, particularly as the above mode of demonstrating this principle

6th. If three terms of a proportion be given, the fourth term can easily be found. Let there be the three terms of a proportion,

$$AB : a\,b = AC :$$

to which the fourth term is wanting. Then, by knowing how many times the line *a b* is smaller than the line AB, or, which is the same, whatever part of the line AB, the line *a b* is; you can easily take the same part of the line AC, which will be the fourth term of your proportion. If you know, for instance, that the line *a b* is one half of the line AB, you would at once conclude, that the required fourth term in your proportion must be one half of the line AC: this is, as we know, really the case with our proportion, where the fourth term *a c*, which we supposed here to be unknown, is really one half of AC. If *a b* were one third of AB, you would conclude that your fourth term must be one third of AC; and so on. If, instead of the fourth term, another, for instance the second

admits of an ocular demonstration by measurements. For if the teacher uses *lines* for the terms of his proportions, and not abstract numbers, which are always more difficult to be comprehended, he can actually perform these additions, by extending the line AB, for instance, to once or twice the length of the line *a b*, and then show, by measuring these lines, that the first term is really as many times greater than the second term, as the third term is greater than the fourth term. In this manner the demonstrations will not only be perfectly geometrical, but also have the advantage of the inductive method.

term, were unknown, you could find it in a manner similar to the one just given. For one ratio being expressed, you will always know the relation which the term you are to find must bear to the term with which it is to form a ratio. In our example, the ratio, AB to *a b*, is expressed; and consequently also the relation of the unknown term to the term AC, with which it is to form the second ratio. We know that the unknown term must be one half of AC, and that is sufficient to complete our proportion.

7th. Geometrical proportions are also frequently made use of in common Arithmetic, and in Algebra. You can say of the two numbers 3 and 6, that they are in proportion to the numbers 4 and 8; because 3 are as many times contained in 6, as 4 in 8, which may be expressed thus:

$$3 : 6 = 4 : 8.$$

Thus, if four lines are, together, in a geometrical proportion, their length, expressed in numbers of rods, feet, or inches, will be in the same proportion.

8th. It is to be remarked, that in every geometrical proportion, expressed in numbers,* the product obtained by multiplying the two mean terms together, is equal to the product† obtained

* For we cannot multiply *lines* together, but merely the *abstract numbers*, which express their relative length.

† When you multiply one number by another, the answer is called the product of those numbers.

by multiplying the two extreme terms together. In the above proportion, $3 : 6 = 4 : 8$, for instance, we have 3 times 8, equal to 6 times 4. For the first of our extreme terms 3, being exactly as many times *smaller* than the first of our mean terms 6, as the second of our extreme terms 8, is *greater* than the second of our mean terms 4, (namely, twice): what the multiplier 3, in the one case, is *smaller* than the multiplier 6 in the other, is made up by the multiplicand 8, which is as many times *greater* than the multiplicand 4, as the multiplier 6 than the multiplier 3; and in a similar manner we could prove the same of any other geometrical proportion. To give but one more example: In the proportion

$$2 : 8 = 3 : 12,$$

we have, again, twice 12 equal to 8 times 3; because the first multiplier 2 is exactly as many times *smaller* than the second multiplier 8, as the first multiplicand 12 is *greater* than the second multiplicand 3 (namely, here 4 times). If the ratios of our proportion were inverted, as for instance, $8 : 2 = 12 : 3$, our principle would still prove to be correct. For we have again 8 times 3, equal to twice 12. The only difference consists in the mean terms having now become the extreme terms, and vice versa. If we change the order of our ratios, namely,

$$12 : 3 = 8 : 2,$$

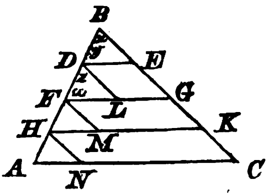
the result of the multiplication of the mean and

extreme terms is still the same. For 3 times 8 are the same as 8 times 3; and 12 times 2 the same as twice 12.

* * *

What you have now learned of geometrical proportions will enable you to understand every principle in plane Geometry; we will therefore continue our inquiries into the principles of geometrical figures.

QUERY XIV.



If you divide one side, AB, of a triangle ABC, into any number of equal parts, four for instance, and then, from the points of division D, F, H, draw the lines, DE, FG, HK, parallel to the side AC, what remark can you make with regard to the other side, BC?

A. That the other side, BC, will also be divided into as many equal parts as the side AB.

Q. How do you prove this?

A. By drawing the lines DL, FM, HN, parallel to the side BC, the triangles, BDE, DFL, FHM, HAN, will all be equal to one another. For

comparing, in the first place, the two triangles, BDE, DFL, we see that the side BD is equal to the side DF; (because we have divided the line AB into equal parts); and the angle x is equal to the angle z; because these angles are formed by the two parallels, DL and BC, being intersected by the straight line AB (Query 11, Sect. I.); and the angle y is equal to the angle w; because y and w, are formed, in a similar manner, by the two parallels, DE, FG, being intersected by the same straight line AB: therefore, because we have one side DB, and the two adjacent angles x and y, in the triangle BDE, equal to one side DF, and the two adjacent angles, z and w, in the triangle DFL, these two triangles must be equal to each other (Query 6, Sect. I.); and the side DL, opposite to the angle w, in the triangle DFL, must be equal to the side BE, opposite to the equal angle y, in the triangle BDE; and in the same manner it can be proved, that FM and HN, are also equal to BE. Now, each of the quadrilaterals, DELG, FGMK, HKNC, is a parallelogram, (because the opposite sides have been drawn parallel); and as the opposite sides of a parallelogram are equal (page 59, 1st), DL must be equal to EG, FM to GK, and HN to KC. But each of the lines, DL, FM, HN, is equal to BE; therefore, each of the lines, EG, GK, KC, must also be equal to BE; that is, the line BC will be

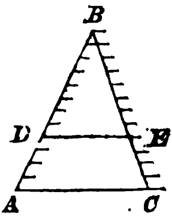
divided into the same number of equal parts as the line AB.

Q. Will you now prove the same principle in the case where the line AB is divided into five, six, or more equal parts?

* * *

QUERY XV.

If in a triangle, ABC, you draw a line, DE, parallel to one of the sides, AC for instance, what relation must the parts, BD, DA; BE, EC, into which the sides, AB and BC, are divided, bear to one another, and to the sides AB, BC, themselves?



A. The upper parts, BD and BE, as well as the lower parts, DA and EC, will be to each other in the same ratio, in which the whole sides AB, BC, themselves are.

Q. Why?

A. Because you can imagine the side AB to be successively divided into smaller and smaller parts, until one of the points of division shall have fallen upon the point D. Then, by drawing, through all the points of division, parallels to the side AC, the side BC will be divided into as many equal parts as the side BA (last Query); and as the line DE will, itself, be one of these parallels, BE will have as many of these parts

marked as BD; and EC as many as DA: and, therefore, the ratio of the *whole* of the line BA to the *whole* of the line BC, must be the same as that of BD to BE, or DA to EC.

Q. How can you express these proportions in writing?

A. $$BA : BC = BD : BE$$
$$BA : BC = DA : EC;$$

consequently also

$$BD : BE = DA : EC$$

(3d principle of proportion).

Q. Would the reverse of the same principle be also true; that is, must the line DE always be parallel to AC, when the parts BD and BE, and DA and EC, are proportional to each other, or to the whole of the sides, BA, BC?

A. Yes. For you need only imagine the side BA to be again successively divided into smaller and smaller parts, until one of the points of division shall have fallen upon D. Then, it is evident, that by drawing, as before, through the points of division, parallels to the side AC, DE itself, must be one of them; for the point E must necessarily be a point of division on the side BC, and the corresponding one to D on the side AB, if BE shall again have as many of these parts marked as BD; and EC as many as DA; and only in this case can the ratio of BE to BD, and EC to DA, be the same as the ratio of the whole side BC, to the whole side BA; that is, only in this case can BE, EC, and BD, DA, be

proportional to each other, and to the whole of the sides BC and BA.

Remark. It has already been stated (Theory of proportions, page 63), that two geometrical figures cannot be similar to each other, unless they are constructed after the same manner, and have their sides proportional. We will now give the strictly geometrical definition of the same principle for *rectilinear* figures.

In order that two rectilinear figures may be similar to each other, it is necessary,

*1st, That both figures should be composed of the same number of sides;**

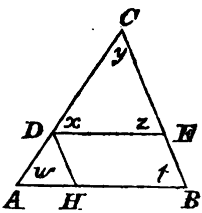
2dly, That the angles in one figure should be equal to the angles in the other, respectively;

3dly, That these angles should follow each other in precisely the same order in both figures; and

4thly, That the sides, which include the equal angles in both figures, (and which are therefore called the corresponding or homologous sides†), should be in a geometrical proportion.

QUERY XVI.

If in a triangle, ABC, you draw a line DE, parallel to one of the sides, AB, what relation will the triangle DEC, which is cut off, bear to the whole of the triangle ABC?



A. The triangle DEC will be similar to the triangle ABC.

* This will, of course, always be the case in triangles.

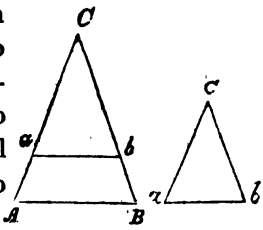
† In triangles, the corresponding sides are those which are opposite to the equal angles.

Q. Why?

A. Because the three angles, x, y, z, of the triangle DEC, will be equal to the three angles w, y, t, of the triangle ABC, each to each; for the angles x and z, are respectively equal to the angles w, t; because the line DE is parallel to AB (Query 11, Sect. I.). Moreover, we have the proportion CD : CE = CA : CB (preceding Query), and by drawing DH parallel to the side CB, also CD : BH (or ED) = AC : AB; and, therefore, the three sides of the triangle DEC, are proportional to the three sides of the triangle ABC: consequently these two triangles must be similar to each other.

QUERY XVII.

If the three angles in one triangle are equal to the three angles in another triangle, each to each, what relation will these triangles bear to each other?



A. They will be similar?

Q. How can you prove it?

A. By applying the triangle $a\,b\,c$ to the triangle ABC, the angle at c will coincide with the angle at C, and the side $c\,a$ will fall upon CA, and $c\,b$ upon CB; and as the angles at a

and *b*, in the triangle *a b c*, are respectively equal to the angles at A and B, in the triangle ABC, the side *a b* will fall parallel to the side AB; because the two lines, *a b* and AB, will be cut by the straight line AC, at equal angles (Query 9, Sect. I.), and we shall have the same case as in the preceding Query: consequently, the triangles, *a b c* and ABC, will be similar to each other.

Q. Supposing you have a triangle, in which you know but two angles respectively equal to two angles in another triangle, what can you infer with regard to these two triangles?

A. That they must still be similar to each other. For two angles of a triangle always determine the third one* (page 37,—2.)

QUERY XVIII.

If you have two triangles, a b c, *ABC* (see the last figure), *and only know that one angle at* c, *in the one, is equal to one angle at C in the other, but that the sides, which include that angle in both triangles, are in a geometrical proportion, what inference can you draw from it?*

A. That these triangles will again be similar to each other. For if you imagine the triangle

* The third angle in every triangle, making with the two other angles two right angles, will always be equal to two right angles less the sum of the two other angles.

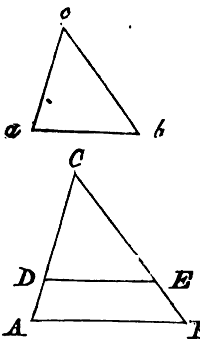
a b c, placed as before, upon the triangle ABC, the angle at *c* will again coincide with the angle at C, and the side *c a* will fall upon CA, and *c b* upon CB; and as *c a* and *c b* are proportional to CA and CB, the side *a b* must be parallel to the side AB (Query 15, Sect. II.); and we shall have the same case as in Query 15, Sect. II.; consequently the triangle *a b c* will be similar to the triangle ABC.

QUERY XIX.

Let us now consider the case, where all the angles of two triangles are unknown; but the three sides of the one are in proportion to the three sides of the other; what relation will these triangles bear to each other?

A. They will be similar to each other.

Q. How can you prove it?



A. Let us suppose, for instance, that the three sides of the triangle *a b c*, are in proportion to the three sides of the triangle ABC; that is, let us have the proportions

$$a c : a b = AC : AB$$
$$a c : c b = AC : CB.$$

Then make CD equal to *c a*, and draw through the point D, the line DE parallel to AB; and the triangle

CDE will be similar to the triangle CAB (Query 16, Sect. II.), and we shall have the proportions

$$DC : DE = AC : AB$$
$$DC : CE = AC : CB,$$

in which the two ratios, AC : AB, and AC : CB, are the same as in the first two proportions (page 85); consequently, by combining these two proportions with the two preceding ones, we shall have

$$DC : DE = ac : ab$$
$$DC : CE = ac : ab;$$

(see theory of proportions, principle 3d).

Now, as I have made DC equal to ac, I can write ac instead of DC, in the two last proportions; and they will then become

$$ac : DE = ac : ab,$$
$$ac : CE = ac : cb.$$

The upper one expresses, that the line DE is as many times greater than the line ac, as the line ab is greater than the same line ac (Definition of geometrical proportions); consequently the line DE is equal to the line ab. In like manner the lower one expresses, that the line CE is as many times greater than the line ac, as the line cb is greater than the same line ac; consequently the line CE is equal to the line cb; therefore the three sides of the triangle DEC, are equal to

the three sides of the triangle *a b c*, each to each, viz :

The side DC = the side *a c*
" " DE = " " *a b*
" " CE = " " *c b*;

therefore these two triangles are equal to one another (Query 4, Sect. II.); and as the triangle DEC is similar to the triangle ABC, the triangle *a b c* will also be similar to the triangle ABC.

*
* *

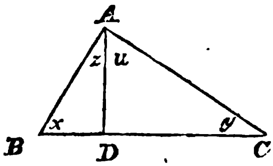
Q. Will you now briefly state the different cases, in which two triangles are similar to one another?

A. 1st, When the three angles in one triangle are equal to the three angles in another, each to each; and also when two *angles in one triangle are equal to two angles in another, each to each; because then the third angle in the one will be equal to the third angle in the other.*

2dly, When an angle in one triangle is equal to an angle in another, and the two sides which include that angle in the one triangle, are in proportion to the two sides which include that angle in the other triangle.

3dly, When the three sides of the one triangle are in proportion to the three sides of the other.

QUERY XX.



If you have a right-angular triangle ABC, and from the vertex A, of the right angle, drop a perpendicular AD, upon the hypothenuse BC, what relation will the two triangles, ABD and ACD, into which the whole triangle will be divided, bear to each other, and to the triangle ABC itself?

A. The two triangles, ABD and ACD, will be similar to each other, and to the whole triangle ABC.

Q. How do you prove this?

A. The triangle ABD is similar to the whole triangle ABC, because the two triangles being both *right-angular*, and having the angle at x common, will have two angles in one triangle, respectively equal to two angles in the other (page 87, case 1st); and for the same reason is the triangle ACD similar to the whole triangle ABC; (both being *right-angular*, and having the angle y common); and as each of the two triangles, ABD, ACD, is similar to the whole triangle ABC, these two triangles must also be similar to each other. (Truth I.)

Q. What important inferences can you draw from the principle you have just established?

A. 1st, In the two similar triangles, ABD and ACD, the sides which are opposite to the equal angles, must be in proportion (condition 3d of geometrical similarity, page 82); and we shall therefore have the proportion

$$BD : AD = AD : DC;^*$$

that is, *the perpendicular AD, is a mean proportional*† *between the two parts into which it divides the hypothenuse.*

2dly, From the two similar triangles, ABC, ABD, we shall have the proportion

$$BD : AB = AB : BC;$$

that is, *the side AB, of the right-angular triangle ABC, is a mean proportional between the whole hypothenuse BC, and the part BD, cut off from it by the perpendicular BD.*‡

*The first ratio is formed by the two sides, BD and AD, of the triangle ADB, of which BD is opposite to the angle z, and AD to the angle x. The second ratio is formed by the two corresponding sides, AD, DC, of the triangle ADC, in which the side AD is opposite to the angle y, which is equal to the angle z, to which the side BD is opposite in the triangle ADB; and the side DC is opposite to the angle u, which is equal to the angle x, to which the side AD is opposite in the triangle ADB.

†When, in a geometrical proportion, the two mean terms are equal to one another, either of them is called a mean proportional between the two extremes.

‡The part BD of the hypothenuse, situated between the extremity B of the side AB, and the foot D of the perpendicular AD, is sometimes called the *adjacent segment* to AB. (Legendre's Geometry, translated by Professor Farrar.)

3dly, The two similar triangles, ABC and ACD, give the proportion

$$DC : AC = AC : BC;$$

that is, ***the other side AC of the right-angular triangle ADC, is also a mean proportional between the whole hypothenuse and the other part DC, cut off from it by the perpendicular AD.***

Remark. The five last queries comprise one of the most important parts of Geometry. The principles contained in them are applied in the solution of almost every geometrical problem; the beginner will therefore do well to render himself perfectly familiar with them.

RECAPITULATION OF THE TRUTHS CONTAINED IN THE SECOND SECTION.

PART I.

Quest. Can you now repeat the different principles respecting the equality and similarity of triangles, which you have learned in this section?

Ans. 1. If in two triangles two sides of the one are equal to two sides of the other, each to each, and the angles which are included by them also equal to one another, the two triangles must be equal in all their parts, that is, they must coincide with each other throughout.

2. In equal triangles, that is, in triangles which coincide with each other, the equal sides are opposite to the equal angles.

3. If one side and the two adjacent angles in one triangle, are equal to one side and the two adjacent angles in another triangle, each to each, the two triangles are equal, and the angles opposite to the equal sides are also equal.*

4. The two angles at the basis of an isosceles triangle are equal to one another.

5. If the three sides of one triangle are equal to the three sides of another, each to each, the two triangles coincide with each other throughout; that is, their angles are also equal, each to each.

6. In every triangle the greater side is opposite to the greater angle, and the greatest side to the greatest angle; and the reverse is also true, namely: the greater angle is opposite to the greater side, and the greatest angle to the greatest side.

7. In a right-angular triangle the greatest side is opposite to the right angle.

8. When a triangle contains two equal angles, it also has two equal sides, and the triangle is isosceles.

* This principle, though already demonstrated in the first section, is repeated here, in order to complete what is said on the equality of triangles.

9. If the *three* angles in a triangle are equal to each other, the sides are also equal, and the triangle is equilateral.

10. Any one side of a triangle is smaller than the sum of the two other sides.

11. If from a point within a triangle two lines are drawn to the two extremities of one of the sides of the triangle, the angle made by those lines is always greater than the angle of the triangle which is opposite to that side; but the sum of the two lines, which make the interior angle, is *smaller* than the sum of the two sides which include the smaller angle of the triangle.

12. If from a point without a straight line a perpendicular is dropped upon that line, and at the same time other lines are drawn obliquely to different points in the same straight line, the perpendicular is shorter than any of the oblique lines, and is therefore the shortest line that can be drawn from that point to the straight line.

13. The distance of a point from a straight line is measured by the length of the perpendicular dropped from that point to the straight line.

14. Of several oblique lines drawn from a point without a straight line to different points in that straight line, that one is the shortest, which is nearest the perpendicular, and that

one is the greatest, which is furthest from the perpendicular.

16. If a perpendicular is drawn to a straight line, two oblique lines drawn from two points in the straight line, on each side of the perpendicular and at equal distances from it, to any point in that perpendicular, are equal to one another.

17. If a perpendicular is drawn to a straight line, there is but one point in the straight line on each side of the perpendicular such, that a straight line drawn from it to a given point in that perpendicular, is of a given length.

18. If a perpendicular is drawn to a straight line, there is but one point in the straight line, on each side of the perpendicular, from which a line drawn to a given point in that perpendicular, makes with the straight line an angle of a given magnitude.

19. If two sides and the angle which is opposite to the greater of them in one triangle, are equal to two sides and the angle which is opposite to the greater of them in another triangle, each to each, the two triangles coincide with each other in all their parts; that is, they are equal to each other.

20. If the hypothenuse and one side of a right-angular triangle are equal to the hypothenuse and one of the sides of another right-angular triangle, each to each, the two right-angular triangles are equal.

21. If in two triangles two sides of the one are equal to two sides of the other, each to each, but the angle included by the two sides in one triangle is greater than the angle included by the two sides in the other, the side opposite to the greater angle in the one triangle is greater than the side opposite to the smaller angle in the other triangle.

22. If in a parallelogram a diagonal is drawn, it divides the parallelogram into two equal triangles.

23. The opposite sides of a parallelogram are equal to each other.

24. The opposite angles in a parallelogram are equal to each other.

25. By *one* angle of a parallelogram the *four* angles are determined.

26. A quadrilateral, in which the opposite sides are respectively equal, is a parallelogram.

27. A quadrilateral, in which two sides are equal and parallel, is a parallelogram.

28. If from one of the vertices of a rectilinear figure, diagonals are drawn to all the other vertices, the figure is divided into as many triangles as it has sides less two.

29. The sum of all the angles in a rectilinear figure, is equal to as many times two right angles as the figure has sides less two.

* * *

RECAPITULATION OF THE TRUTHS CONTAINED IN PART II.

1. *On Proportions.*

Ques. 1. How is a geometrical ratio determined?

Q. 2. What is the ratio of a line 3 inches in length to a line of 12 inches? What the ratio of a line 2 inches in length to one of 10 inches? &c.

Q. 3. When two geometrical ratios are equal to one another, what do they form?

Q. 4. What is a geometrical proportion?

Q. 5. What signs are used to express a geometrical proportion?

Q. 6. What sign is put between the two terms of a ratio?

Q. 7. What sign is put between the two ratios of a proportion?

Q. 8. What are the first and fourth terms of a geometrical proportion called?

Q. 9. What are the second and third terms of a geometrical proportion called?

Q. 10. What are the most remarkable properties of geometrical proportions?

Ans. a. In every geometrical proportion the two ratios may be inverted.

b. In every geometrical proportion the order of the means or extremes may be inverted.

c. If two geometrical proportions have a ratio common, the two remaining ratios make again a proportion.

d. If you have several geometrical proportions, of which the second has a ratio common with the first, the third a ratio common with the second, the fourth a ratio common with the third, &c., the sum of all the first terms will be in the same ratio to the sum of all the second terms as the sum of all the third terms is to the sum of all the fourth terms; that is, the sums make again a proportion.

e. The second term of a proportion being added once, or any number of times, to the first term, and the fourth term the same number of times to the third term, they will still be in proportion; and in the same manner can the first term be added a number of times to the second term and the third the same number of times to the fourth term, without destroying the proportion.

f. From three terms of a geometrical proportion the fourth term can be found.

g. If four lines are together in a geometrical proportion, their lengths expressed in numbers of rods, feet, or inches, &c. will be in the same proportion.

h. In every geometrical proportion the product obtained by multiplying the two mean terms together is equal to the product obtained by multiplying the two extreme terms together.

Quest. How can you prove each of these principles?

* * *

QUESTIONS ON SIMILARITY OF TRIANGLES.

Quest. What other principles do you recollect to have learned in the second part of the 2d section?

Ans. 1. If one side of a triangle is divided into any number of equal parts, and then, from the points of division, lines are drawn parallel to one of the two other sides, the side opposite to the one that has been divided will by these parallels be divided into as many equal parts as the first side.

2. If, in a triangle, a line is drawn parallel to one of the sides, that parallel divides the two other sides into such parts as are in proportion to each other and to the whole of the two sides themselves; and the reverse of this principle is also true; namely, a line must be parallel to one of the sides of a triangle, if it divides the two other sides proportionally.

3. If, in a triangle, a line is drawn parallel to one of the sides, the triangle which is cut off by it, is similar to the whole triangle.

4. If the three angles in one triangle are equal to the three angles in another triangle, each to each, the two triangles are similar to one another; and the same is the case if only *two* angles in one triangle are equal to two angles in another, each to each.

5. If an angle in one triangle is equal to an angle in another, and the two sides which include that angle in the one triangle are in proportion to the two sides which include the equal angle in the other triangle, these two triangles are similar to each other.

6. If the three sides of a triangle are in proportion to the three sides of another triangle, these two triangles are similar to each other. *

7. If, in a right-angular triangle, a perpendicular is dropped from the vertex of the right angle upon the hypothenuse, that perpendicular divides the whole of the triangle into two triangles, each of which is similar to the whole triangle, and which are consequently similar to each other.

8. The perpendicular dropped from the vertex of a right-angular triangle to the hypothenuse is a mean proportional between the parts into which it divides the hypothenuse.

9. In every right-angular triangle, each of the sides which include the right angle is a

* The teacher will do well to let the pupil repeat the different cases where two triangles are similar to each other. (page 87.)

mean proportional between the hypothenuse and that part of it, which lies between the extremity of that side and the foot of the perpendicular dropped from the vertex of the right angle upon the hypothenuse. *

---

* The teacher may now ask his pupils to demonstrate these principles.

SECTION III.

OF THE MEASUREMENT OF SURFACES.

Preliminary Remarks. We determine the length of a line, by finding how many times another line, which we take for the measure, is contained in it. The line which we take for the measure, is chosen at pleasure; it may be an inch, a foot, a fathom, a mile, &c. If we have a line upon which we can take the length of an inch 3 times, we say that line measures 3 inches, or is 3 inches long. In like manner, if we have a line upon which we can take the length of a fathom 3 times, we call that line 3 fathoms, &c. To find out which of two lines is the greater, we must measure them. If we take an inch for our measure, that line is the greater, which contains the greater number of inches. If we take a foot for our measure, that line is the greater, which contains the greater number of feet, &c.

To measure the extension of a surface, we make use of another surface, commonly a square (□), and see how many times it can be applied to it; or, in other words, how many of those squares it takes to cover the whole surface. The length of the square side is arbitrary. If it is an inch, the square of it is called a *square inch;* if it is a foot, a *square foot;* if it is a mile, a *square mile,* &c. The extension of a surface, expressed in numbers of square miles, rods, feet, inches, &c., is called the *area* of the surface.

Remark 2. If we take one of the sides of a triangle for the *basis;* the perpendicular dropped from the vertex of the *opposite angle* to that side is called the *altitude* or *height* of the triangle.

If in the triangle ABC, (Fig. I.) for instance, we call AC the basis the perpendicular BD will be its height (altitude). If the perpendicular BD should fall without the trian-

Fig. I.

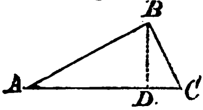
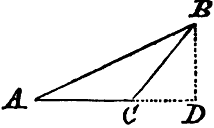
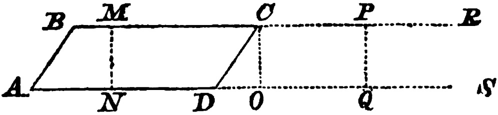


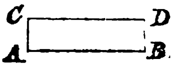
Fig. II.



gle ABC, (as in Fig. II.) we need only extend the basis, and then drop the perpendicular BD upon the farther extension (CD) of the basis AC.



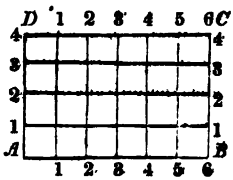
If in a parallelogram ABCD, we take AD for the basis, any perpendicular, MN, CO, PQ, &c. dropped from the opposite side BC, or its farther extension CR upon that basis, or its farther extension DS, will measure the height of the rectangle. For in a parallelogram the opposite sides are parallel to each other (see Definitions). and all the perpendiculars, dropped from one of two parallel lines to the other, are equal (Query 12. Sect. I.) What in this respect holds of a parallelogram is applied also to a square, a rhombus, and a rectangle; for these three figures are only modifications of a parallelogram.—(See Definitions.)



As in a rectangle ABCD, the adjacent sides AB, BD, are *perpendicular* to each other, it is evident that if AB is taken for the basis, the side BD itself will be the height of the rectangle.

Remark III. We call two geometrical figures equal * to one another, when they have equal areas, (see preliminary remark to sect. II.)—Thus a *triangle* is said to be equal to a *rectangle* when it contains the same number of square miles, rods, feet, inches, &c. as that rectangle.

QUERY I.



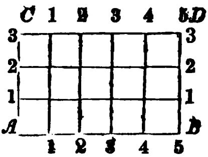
If the basis AB, of a rectangle ABCD, measures 6 *inches, and the height, the side BC,* 4 *inches, how many square inches are there in the rectangle?*

* The term *equivalent* would undoubtedly be better; but as there is no generally adopted sign in Mathematicks to express, that two

A. Twenty-four.

Q. How can you prove this?

A. If a rectangle is 4 inches high, I can divide it like the rectangle ABCD (see the figure) into four rectangles each of which shall be *one* inch high, and have its basis equal to the basis of the whole rectangle. And as the basis AB, of the rectangle measures 6 inches, by raising upon it, at the distance of an inch from each other, the perpendiculars 1, 2, 3, 4, 5, each of the four rectangles will be divided into 6 square inches; and therefore the whole rectangle ABCD into 24 square inches.



Q. *How many square inches are there in a rectangle, whose basis is* 5, *and height* 3 *inches?*

A. Fifteen. Because in this case I can divide the rectangle into 3 rectangles of 5 square inches each.

Q. Supposing the measurements of the first rectangle (see the 1st figure) were given in feet, in rods, or in miles, instead of inches, how many square feet, rods or miles would there be in the rectangle?

A. If its measurements were given in feet, it would contain 24 square feet; if they were

things are equivalent without being exactly the same, we are obliged to use the sign *equal.*

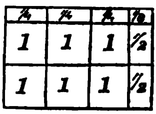
given in rods, it would contain 24 square rods; and if in miles, 24 square miles; for in these cases I need only imagine the lines, 1, 2, 3, 4, &c. to be drawn a foot, a rod, a mile apart; the *number* of divisions will remain the same; nothing but their size will be altered.—And in the same manner; if the measurements of the second rectangle were given in feet, rods, or miles, it would contain 15 square feet, rods, or miles &c.*

Q. Can you now give a general rule for finding the area of a rectangle?

A. Yes. Multiply the length of the basis, given in miles, rods, feet, inches, &c. by the height expressed in units of the same kind.

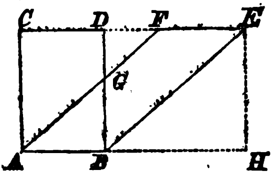
Q. Can you now tell me how to find the area of a square?

A. The area of a square is found by multiplying one of its sides by itself.—For a square is a rectangle whose sides are all equal (see Definitions) and the area of a rectangle is found by multiplying the basis by one of the two adjacent sides.



* The teacher may also give his pupils a rectangle, whose measurements are both given in fractions; for instance, a rectangle of $3\frac{1}{2}$ inches in length and $2\frac{1}{4}$ inches high, and then shew by the figure that this rectangle will measure 6 square inches, two half square inches, $\frac{3}{4}$ and $\frac{1}{8}$ of a square inch, in the whole $7\frac{7}{8}$ square inches, which is the answer to the multiplication of $3\frac{1}{2}$ by $2\frac{1}{4}$.

QUERY II.



If a parallelogram ABEF stands on the same basis, AB, as a rectangle, ABCD, and has its height equal to the height of that rectangle, what relation do the areas of these two figures bear to each other?

A. The area of the parallelogram ABEF is equal to the area of the rectangle ABCD; therefore I can say that the parallelogram ABEF is equal to the rectangle ABCD, (see remark 3d. Introd. to Sect. III.)

Q. How do you prove it?

A. The right-angular triangle ACF, has the hypothenuse AF and the side AC equal to the hypothenuse BE and the side BD in the right-angular triangle BDE, each to each, (AF and BE, AC and DB, being opposite sides of the parallelogram ABEF, and the rectangle ABCD); therefore these two triangles are equal (page 55); and by taking from each of the two equal triangles ACF, BDE, the part DGF common to both, the remainders AGDC, BGFE, are also equal (truth IV); and then by adding again to each of the equal remainders the same triangle ABG, the sums, that is, the rectangle ABCD

and the parallelogram ABEF are also equal to one another, (truth III.)

Q. What important truths can you infer from the one you have just demonstrated?

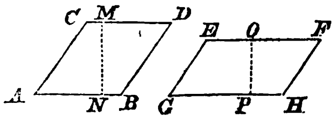
A. 1*st. All parallelograms, which have equal bases and heights, are equal to one another; for each of them is equal to a rectangle upon the same basis and of the same height.* (Truth 1.)

2*dly. Parallelograms upon equal bases and between the same parallels are equal to one another; for if they are between the same parallels their heights must be equal. (Query* 12. *Sect. I.)*

3*dly. The area of a parallelogram is found by multiplying the basis, given in rods, feet, inches, &c., by the height, expressed in units of the same kind. Because the area of the rectangle upon the same basis and of the same height to which it is equal, is found in the same manner.**

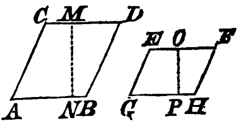
4*thly. The areas of parallelograms are to each other, as the products obtained by multiplying the length of the bases of the parallelograms by their heights, because these products* are *the areas of the parallelograms.*

* The area of a rhombus or lozenge is found like that of a parallelogram; a lozenge being only a peculiar kind of parallelogram.



The parallelogram ABCD, for instance, is to the parallelogram GHEF, as the answer obtained by multiplying the length of the basis AB, by the height MN, is to the answer obtained by multiplying the length of the basis GH, by the height OP; because AB multiplied by MN is the area of the parallelogram ABCD, and GH multiplied by OP is the area of the parallelogram GHEF. This proportion may be expressed thus:

$$\text{Parallelogram ABCD} : \text{parallelogram GHEF} = AB \times MN : GH \times OP.$$



5thly. Rectangles or parallelograms, which have equal bases, are to each other as their heights. For if in the above proportion the basis AB is equal to the basis GH, I can write AB instead of GH, and thereby change it into

$$\text{Parallelograms ABCD} : \text{parallelograms GHEF} = AB \times MN : AB \times OP,$$

that is, the parallelogram ABCD is to the parallelogram GHEF, as AB times the height MN is to AB times the height OP; or, which

is the same, as the height MN alone is to the height OP alone; * which is written thus:

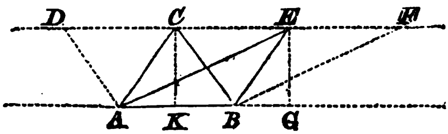
Parallelogram ABCD : parallelogram GHEF
=MN : OP.

6thly. *In precisely the same manner it may be proved, that if the heights MN and OP are equal to each other, the parallelograms, ABCD, GHEF, are to each other as their bases AB and GH; which may be expressed thus:*

Parallelogram ABCD : parallelogram GHEF
=AB : GH.

QUERY III.

If two triangles ABC, ABE, stand on the same basis AB, and have equal heights CK, EG, what relation do the areas of these triangles bear to each other?



A. The areas of these triangles are equal to one another.

* For let the length of the basis AB be expressed by any number you please, 10 for instance; then it is evident that ten times the height MN is in the same ratio to ten times the height OP, as the height MN alone is to the height OP alone. The ratio of 3 to 6, for instance, is as 1 to 2; because 3 is twice contained in 6; and the ratio of 10 times 3 to 10 times 6 (30 to 60) is also as 1 to 2; and so is the ratio of 20 times 3 to 20 times 6 (60 to 120) as 1 to 2; the ratio of 50 times 3 to 50 times 6 (150 to 300) as 1 to 2, and so on.

Q. How can you prove it?

A. Draw the line AD parallel to CB; BF parallel to AE; and through the two vertices C and E, the line CF parallel to AG (which is possible since the heights CK and EG are equal) and the area of the parallelogram ABCD will be equal to the area of the parallelogram ABEF (query 2, Sect. IV.); and as the triangle ABC is half of the parallelogram ABCD—for the diagonal AC divides the parallelogram ABCD into two equal parts (query 12, Sect. II.)—and the triangle ABE is half of the parallelogram ABEF; therefore the areas of these two triangles must also be equal to one another; for if the *wholes* are equal, the *halves* are also equal; and the same can be proved of triangles, which have *equal* bases and heights.

Q. What consequences follow from the principle just advanced?

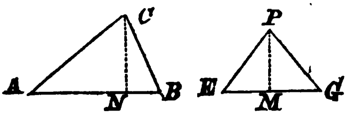
A. 1st. *Every triangle is half of a parallelogram upon equal basis and of the same height.* (This is evident from looking at the figure, and from Query 12, Sect. II.)

2d. *The* area *of a triangle is half of the arèa of a parallelogram upon the same basis and of the same height. Thus the area of a triangle is found by multiplying the length of the basis by the height, and dividing the product*

by 2;* for the area of a parallelogram is equal to the *whole* product of the length of the basis multiplied by the height.†

3d. *The areas of triangles upon the same basis and between the same parallels are equal;* because if they are between the same parallels their heights must be equal; and we have the same case as in the last query; namely triangles upon the same basis and of equal heights.

4th. *The areas of triangles are to each other as the products obtained by multiplying the length of their bases by their heights;* for these products *are* the areas of the triangles. Thus



the area of the triangle ABC is to the area of the triangle EGP, as the length of the basis AB multiplied by the height CN, is to the length

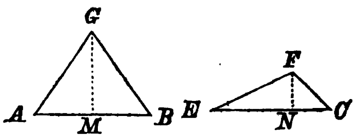
* Instead of multiplying the basis by the whole height and dividing the product by 2; you may simply multiply the basis by half the height, or the height by half the basis.

† If the basis of a triangle is 8 feet and the height 4 feet, the area of the triangle is equal to 32 (4 times 8) divided by 2; that is, 16 square feet; whereas the rectangle upon 8 feet basis and 4 feet high measures 32 (4 times 8) square feet, which is exactly double of the area of the triangles.

of the basis EG, multiplied by the height PM; which may be expressed thus:

Triangle ABC : triangle EGP $=$ AB $\times$ CN : EG $\times$ PM.

5thly. *The areas of triangles upon equal bases are to each other as the heights of the triangles;* because the areas of parallelograms upon the same bases and of the same heights are to each other in the ratio of the heights; and their halves (the areas of the triangles) must be in the same ratio.* Thus if the two triangles



ABG, ECF, have their bases AB, EC equal to each other, we have the proportion:

Triangle ABG : triangle ECF $=$ CM : FN.

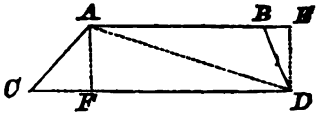
6th. *The areas of triangles, which have equal heights, are to each other as the bases of the triangles.* This truth follows like the preceding one from the same principle established with regard to parallelograms, of which the triangles are the halves. (Page 108, *6thly.*)

* This principle and the following one might have been established immediately from the proportion:

Triangle ABC : triangle EGP $=$ AB $\times$ CN : EG $\times$ PM,

in precisely the same manner as it has been proved for parallelograms. (Page 107, *5thly.*)

QUERY IV.



How do you find the area of a trapezoid?

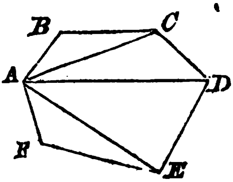
Ans. By multiplying the sum of the two parallel sides by their distance, and dividing the product by 2.

Quest. How can you prove this?

Ans. By drawing the diagonal AD, the trapezoid ABCD, will be divided into the two triangles ACD and ABD. The area of the triangle ACD is found by multiplying the length of the basis, CD, by the height, AF, and dividing the product by 2; (page 109, 2d.) In the same manner we find the area of the triangle ABD, by multiplying the length of the basis, AB, by the height DE, and dividing the product by 2; and as the height, DE, of the triangle ABD, is equal to the height AF, of the triangle ACD, (because DE and AF are perpendiculars between the same parallels) we can find the area of the two triangles, or of the whole trapezoid, ABCD, at once, by multiplying the *sum* of the two parallel lines AB, CD, by their distance AF, (which is the common height of the triangles ACD, ABD,) and dividing the product by 2.*

* If you multiply two numbers successively by the same number, and then add the products together, the answer will be the same

QUERY V.



How do you find the area of a polygon ABCDEF, or, in general, of any other rectilinear figure?

Ans. By dividing it by means of diagonals, (as in the figure before us) or by any other means into triangles. The area of each of these triangles is then easily found by the rule given; (page 109, *2d.) and the sum of the areas of all the triangles, into which the figure is divided, is the area of it.*

---

as the *sum* of the two numbers at once multiplied by that number. If you multiply each of the numbers, 6 and 5, for instance, by 4, and then add the products 24 and 20 together, you will have 44; and adding, in the first place, 6 to 5, and then multiplying the sum, 11, by 4, you will again have 44.

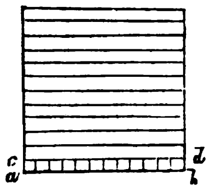
Instead of multiplying the *sum* of the two parallel sides by their distance, and then dividing the product by 2, you may multiply, at once, *half* the sum of the two parallel sides by the distance; or the *sum* of the two parallel sides by *half* their distance.

REMARK.

In calculating the area of geometrical figures, it frequently occurs that the bases and heights of triangles, parallelograms, &c. are given in feet, inches, seconds, thirds, &c. Most treatises on Arithmetic, teach how to perform such multiplications, but not all give a sufficiently clear explanation of the principle on which this operation is founded. It is simply this:

1. If you multiply feet by feet, the product will evidently be square feet; for you may consider the product as the area of a rectangle, whose basis and height are given in feet. (See the figures to Query 1, Sect. III.) For the same reason, if you multiply inches by inches, the product will be square inches; if seconds by seconds, square seconds, and so on.

2. If you multiply feet by inches, each foot multiplied by an inch will give the area of a rectangle *a b c d*, which is one foot (twelve inches) long, and one inch high, which will therefore measure 12 square inches; and it will take twelve of these rectangles to complete a square foot; (see the figure). Thus, if you multiply 3 feet by 6 inches, your product will be 18 such rectangles, or one square foot and a half.



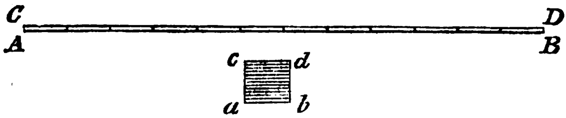
If you multiply feet by seconds, each foot multiplied by one second, will give you the area of a rectangle, which is one foot long, and one second high, and it will therefore take twelve of these rectangles to complete a rectangle a foot long and one *inch* high (because there are twelve seconds in an inch). In the same manner, if you multiply feet by thirds, each foot multiplied by one third, will give you the area of a rectangle which is one foot long and one *third* high; and it will take twelve of these rectangles to complete a rectangle a foot long and one *second* high; (because 12 thirds make a second) and so on.

3. If you multiply inches by seconds, each inch multiplied by one second will give you the area of a rectangle one inch (12 seconds) long and one second high, which will therefore measure twelve square seconds, and it will take twelve of these rectangles

to complete one square *inch.** If you multiply inches by thirds each inch multiplied by one third will give you the area of a rectangle which is one inch long and one *third* high, and it will take twelve of these rectangles to complete a rectangle an inch long and one *second* high, and so on.

4. If you multiply seconds by thirds, each second multiplied by one third, will give the area of a rectangle, which is one second (12 thirds) long and one third high, which will therefore measure twelve square thirds; and it will take twelve of these rectangles to make a square second. If you multiply seconds by fourths, each second multiplied by a fourth will give the area of a rectangle, which is one second long, and one fourth high; and it will take twelve of these rectangles to complete a rectangle which is one second long and one *third* high; (because there are twelve fourths in a third). It is now easy to extend this mode of reasoning farther, to the multiplication of fourths by fifths, &c.

5. It is further important to observe, that each of the rectangles obtained by the multiplication of a *foot* by one *second*, is, in area, equal to a *square inch*. Thus, if you multiply 3 feet by 5 seconds, the product will be 15 square inches. For a rectangle one foot (12 inches) in length, and one second ($\frac{1}{12}$ of an inch) in height, has its basis twelve times greater, and its height twelve times smaller, than an inch; therefore its area is equal to that of a square inch; because what is gained by the greater basis is lost by the smaller height.



If the basis AB, of the rectangle ABCD, represents the length of a foot (12 inches), and the height AD that of a second, you can divide the rectangle ABCD into twelve rectangles, each of which shall be an inch in length and a second in height, and by placing them upon one another, as is done in the figure, you can complete a square inch, *a b c d*. In the same way it may be

* If, in the last figure, *a b* represents the length of an inch, and the height *a c* one second, the rectangle *a b c d*, will measure 12 square seconds.

shown that each of the rectangles obtained by the multiplication of an *inch* by a *third*, is, in area, equal to a *square second*; and so are the rectangles produced by the multiplication of *seconds* by *fourths* equal to *square thirds*, &c.

All this will be plainer by an example.

Suppose you were to find the area of a parallelogram whose basis is 4 feet 10 inches and 6 seconds, and the height 3 feet 4 inches and 7 seconds. In order to perform the multiplication, write the expression for the height under that of the basis, feet under feet, inches under inches, &c. thus:

4°	10′	6″		
3	4	7		
14	7	6		
1	7	6	0	
	2	10	1	6
16	5	10	1	6

Ans. 16 square feet, 70 square inches, and 18 square seconds.

Take, in the first place, the lowest denomination (here seconds) in the multiplicand, and multiply it by the highest denomination (the feet) of the multiplier; the product of 3 feet by 6 seconds is 18 rectangles of a foot in length and one second in height, which, according to page 115, No. 5, are equal to so many square inches. Dividing them by 12, you will have one rectangle of a foot in length and an inch in height, and 6 square inches over. Put down the 6 square inches, and carry the rectangle.

Multiply the 3 feet of the multiplier by the 10 inches of the multiplicand; the product is 30 rectangles of a foot in length and an inch in height, that is, 30 rectangles of 12 square inches each, (page 114, No. 2) and adding the one rectangle you have to carry from the last multiplication, you have 31 of these rectangles, which, divided by 12 (because 12 of them make a square foot) give 2 square feet, and leave 7 rectangles. Put down the 7 rectangles, and carry the 2 square feet.

Multiply 3 feet by 4 feet, the product is 12 square feet, and 2 to carry from the last multiplication, make 14 square feet, which you must put down, feet being the highest denomination you have to multiply with.

Having finished the multiplication by the feet of the multiplier, take the next lower denomination (the inches) of the multiplier, and multiply by it the seconds of the multiplicand; the product of 4 inches by 6 seconds, is 24 rectangles of an inch in length, and a second in height; that is, 24 rectangles, of 12 square seconds each, (page 114, No. 3) which, divided by 12, give 2 square inches, and leave no remainder. Put down the cipher, where the rectangle of 12 square seconds stand, (namely one place farther to the right than the 6, in the first line of the product; because the 6 are square *inches*), and carry the 2 square inches.

Multiply the inches of the multiplicand by the inches of the multiplier; the product of 10 inches by 4 inches, is 40 square inches, (page 114, No. 1) and 2 to carry from the last multiplication, make 42 square inches; which, divided by 12, give 3 rectangles of 12 square inches each, and leave 6 square inches, which you must put down in the column of the square inches; the 3 rectangles are to be carried.

Multiply the feet of the multiplicand by the inches of the multiplier, the product of 4 feet by 4 inches is 16 rectangles of 12 square inches each, (page 114, No. 2,) and the two rectangles from the last multiplication, make 18 of these rectangles, which, divided by 12, give one square foot, and leave 6 rectangles. Put down the 6 rectangles in the column appropriated to them, and the 1 square foot under the 14 square feet.

In precisely the same manner is the third multiplication by the 7 seconds of the multiplier performed. You have only to observe that the square seconds, resulting from the product of the seconds of the multiplicand by the seconds of the multiplier, must be put down one place farther to the right than the cipher which stands in the column of the *rectangle* of 12 square seconds each. It is easy to perceive that this way of multiplying may be extended to thirds, fourths, fifths, &c.*

When the answer is thus obtained in parts, you have only to add the results of the several multiplications, and the sum is the product, that is, the area of the rectangle, which was to be found. In your example you have 16 square feet; 5 rectangles of 12

* This is hardly ever necessary, such small parts becoming imperceptible to our senses.

square inches each, and 10 square inches; one rectangle of 12 square seconds, and 6 square seconds; that is, 16 square feet, 70 square inches, and 18 square seconds.*

For the sake of practice, the learner may multiply

	13°	9′	10″	3‴			
by	6	7	11	5			
	82	11	1	6			
	8	0	8	11	9		
	1	0	8	0	4	9	
			5	9	1	3	3
Answer	92	1	0	3	3	0	3,

or 92 square feet, 12 square inches, 39 square seconds, and 3 square thirds.

	94°	3′	11″	3‴
by	9	1	8	2

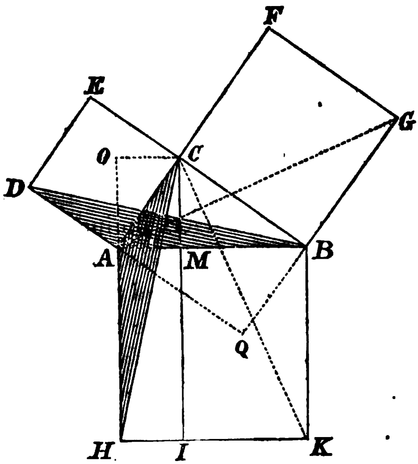
Answer. 862 square feet, 23 square inches, 76 square seconds, and 126 square thirds.

A more tedious way to obtain the answer to the first of the above two sums would be to reduce the multiplicand and multiplier to thirds, and then to take the 23883 thirds of the multiplicand, and multiply them by the 11513 thirds of the multiplier. The product, 274964979 thirds, divided by 144, (for 144 square thirds make 1 square second,) would give 1909479 square seconds, and leave 3 square thirds. Again, 1909479 square seconds, divided by 144, (for 144 square seconds make 1 square inch) give 13260 square inches, and leave 39 square seconds; and finally, 13260 square inches, divided again by 144, (144 square inches making 1 square foot) give 92 square feet and leave 12 square inches; therefore the answer is in both cases the same, namely: 92 square feet, 12 square inches, 39 square seconds, and 3 square thirds; and in the same manner could the answer to the second example be found.

* * *

* 5 rectangles of 12 square inches each, and 10 square inches, making 70 square inches; and 1 rectangle of 12 square seconds, and 6 square seconds, make 18 square seconds.

QUERY VI.



If, upon each of the three sides, AB, AC, BC, of a right-angular triangle ABC, you construct a square, what relation do the squares constructed upon the sides AC, BC, bear to the square constructed upon the hypothenuse, AB?

Ans. The square ABHK, constructed upon the hypothenuse AB, equals, in area, the two squares ACDE, BCGF, constructed upon the two sides AC, BC.

Q. How can you prove it by this diagram, in which the perpendicular CM, is dropped from the vertex C, of the right-angular triangle, ABC, upon the hypothenuse AB, and extended until, in

I, it meets the side HK, opposite to the hypothenuse; and DB and CH are joined?

A. In the first place, I should remark that the two sides AB, AD, of the triangle ABD, are equal to the two sides AH, AC, of the triangle ACH, each to each; (AH and AB, being sides of the same square, ABHK; and, AC, and AD, being sides of the square ACDE) and that the angle DAB, included by the sides AD, AB, is also equal to the angle CAH, included by the two sides AC, AH, (for each of these angles is formed by the angle CAB being added to the right angle of a square); therefore these two triangles are equal to each other. (Query 1, Sect. II.)

Q. Having proved that the triangle ABD is equal to the triangle ACH, what can you infer from it?

A. That the area of the square ACDE, is equal to the area of the rectangle AHIM; for the area of the triangle ABD, is half of the area of the square ACDE, because the triangle ABD stands upon the same basis AD, as the square ACDE, and has its height BQ, equal to the height AC of that square; and I have proved that the area of every triangle is half of the area of a rectangle or square, (a square being nothing but a peculiar kind of rectangle) of equal basis and height. (page 109, 1st.) For the same reason is the area of the triangle ACH, equal to the area of the rectangle AHIM; for the triangle ACH, stands on the same basis AH, as the rectan-

gle AHIM, and has its height CO, equal to the height AM, of that rectangle; and as the *halves*, of the two triangles ABD and ACH, are equal to each other, the *wholes*, the squares ADEC and AHIM, must also be equal to each other; and in precisely the same manner I can prove from the equality of the two triangles ABG and BCK that the square BCFG is equal to the rectangle MBIK; and because the area of the square ADEC, is equal to the area of the rectangle AHIM, and the area of the square BCFG, equal to the area of the rectangle MBIK; therefore the *sum* of the areas of the two rectangles AHIM, and MBIK, that is, the area of the *square upon the hypothenuse AB*, is equal to the sum of the areas of the *squares constructed upon the two sides AC, BC.*

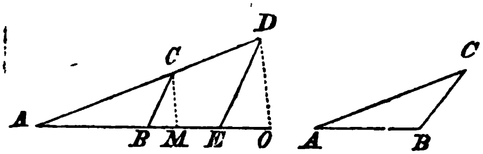
Remark. For the discovery of this principle, we are indebted to Pythagoras, a famous Greek mathematician. It is a very important one, and teaches how to find one of the sides of a right-angular triangle where the two others are given. If, for instance, the two sides AC, BC, of the right-angular triangle ABC, were known to measure, one 5, the other 6 inches, the sum of their squares 25 (5 times 5,) and 36, (6 times 6,) equal to 61, would be the area of the *square* of the hypothenuse; and the square *root* of that number would be the hypothenuse AB, itself. If the hypothenuse and one of the sides are given, you need only *subtract* the square of the side from the square of the hypothenuse, and then the square root of the remainder is the other side. If, for instance, the hypothenuse of a right-angular triangle were 10 feet, and one of the sides 6 feet; the square of the hypothenuse would be 10 times 10, or 100,

and the square of 6, (6 times 6) which is 36, subtracted from 100, leaves 64, which must be the *square* of the side to be found; and taking the square root of it, which is 8, (because 8 times 8 are 64) you will have the *side* itself.*

QUERY VII.

It has been proved (page 110, *4thly*) that the areas of *any* two triangles are to each other as the bases of the triangles multiplied by their heights; can you now find out the proportions in which the areas of two *similar* triangles, for instance the two triangles ABC, AED, are?

A. I should, in the first place, put the triangle ABC, upon the triangle AED, with the



angle at A, upon the angle at A, and from the two vertices C and D, drop the perpendiculars CM, DO, upon AO. Then the two triangles BCM, EDO, are both right-angular, and the angle CBM is equal to the angle DEO; (because in the two similar triangles ABC, AED, the angles ABC, and AED, are equal to each other; and CBM, and DEO, make with them respectively, two right angles,) therefore the

* We shall hereafter give the geometrical solutions of these problems.

third angle CBM in the triangle BCM, will also be equal to the third angle EDO, in the triangle EDO; therefore the two triangles BCM, EDO, are similar (page 87, 1*st.*); and as in similar triangles the sides opposite to the equal angles are proportional, I have the proportion:

$$CM : DO = CB : DE;$$

and because the triangles ABC, ADE, are *supposed* to be similar, I have also the proportion

$$AB : AE = CB : DE.$$

These two proportions have the second ratio common; therefore the two first ratios must again make a proportion, (Theory of Proportions, Principle 3d.) namely:

$$AB : AE = CM : DO.$$

Q. What can you infer from this proportion?

A. That in similar triangles ABC, AED, the bases AB, AE, are to each other as the heights CM, DO, and vice versa.

Now we know that the areas of any two triangles are to each other, as the bases of the triangles multiplied by their heights; therefore between the two triangles ABC, AED, exists the proportion

$$\text{Triangle ABC} : \text{triangle AED} = AB \times CM : AE \times DO;$$

and as these two triangles are similar, we may

in the second ratio ($AB \times CM : AE \times DO$) of the last proportion, write the bases AB, AE, instead of the heights CM, DO, without altering the ratio; for I have just proved that these bases AB, AE, are to each other in the same ratio, as the heights CM and DO; the proportion will then be,

Triangle ABC : triangle AED $=$ $AB \times AB$: $AE \times AE$,

which is read; *the area of the triangle* AED *is as many times greater than the area of the triangle* ABC, *as the side* AE, *multiplied by itself, that is, as the area of the square upon the side* AE, *(page* 104*) is greater than the side* AB *multiplied by itself; or the area of the square upon the side* AB.

Q. Can you prove that the same ratio exists also between the squares upon the two sides AC and AD, and also between the two sides CB and DE, of the similar triangles ABC, AED?

A. Yes. For to prove it of the two sides AC and AD, I need only take *them* for the bases of the two triangles; and to prove it of the sides CB, CD, I must take CB and CD, for the bases; the reasoning would in both cases be the same as that I just went through.

Q. *What would you have obtained, if, in the above proportion,*

Triangle ABC : triangle AED $=$ $AB \times CM$: $AE \times DO$,

you had put the heights CM, DO, instead of the bases AB, AE? *

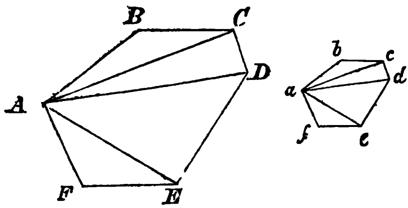
A. The proportion would have been changed into

Triangle ABC : triangle AED = AM × AM : DO × DO;

that is, *the areas of similar triangles are to each other, as the areas of the squares upon the heights of the triangles.*

QUERY VIII.

From the ratio which you have proved to exist between the areas of similar triangles, can you now find out the ratio which exists between the areas of similar polygons? (See Definitions.)



A. Yes. The areas of similar polygons are to each other, as the areas of the squares constructed upon the corresponding sides. The areas of the two similar polygons ABCDEF,

* As the heights of similar triangles are in the same ratio as the bases, you can at pleasure, use one ratio for the other.

a b c d e f, for instance, are to each other, as the areas of the squares constructed upon the sides AB, *a b*, are to the areas of the sides BC, *b c*, &c. For by drawing in the polygon ABCDEF the diagonals AC, AD, AE, and in the polygon *a b c d e f*, the corresponding diagonals *a c*, *a d*, *a e*, the triangle ABC, will be similar to the triangle *a b c*, the triangle ACD, similar to the triangle *a c d*, the triangle ADE similar to the triangle *a d e*, &c.; because, if the whole polygons ABC-DEF, *a b c d e f*, are similar, their similarly disposed parts must also be similar; and the same proportion which exists between their parts, must necessarily exist between the whole polygons; and as the areas of the triangles are in the ratio of the areas of the squares constructed upon the corresponding sides, the whole polygons must be in the same ratio, which may be expressed thus:

Polygon ABCDEF : polygon $a\,b\,c\,d\,e\,f = AB \times AB : a\,b \times a\,b$.

* * *

RECAPITULATION OF THE TRUTHS IN THE THIRD SECTION.

Question 1. How do you determine the length of a line?

2. How do you find out which of two lines is the greater?

3. How do you measure a surface?

4. What do you call the area of a surface?

5. If you take one of the sides of a triangle for the basis, how do you determine the height of the triangle?

6. How is the height of a parallelogram determined? How that of a rectangle? A rhombus? A square?

7. When do you call a triangle equal to a square? To a parallelogram? To a rectangle, &c.?

8. When can you, in general, call two geometrical figures equal to one another, though these figures do not coincide with each other?

9. Will you now repeat the different principles respecting the areas of geometrical figures, which you have learned in this section?

Ans. 1. The area of a rectangle is found by multiplying the length of the basis, given in miles, rods, feet, inches, &c. by the height, expressed in units of the same kind.

2. The area of a square is found by multiplying one of its sides by itself.

3. If a parallelogram stands on the same basis as a rectangle, and has its height equal to the height of that rectangle, the area of the parallelogram is equal to the area of the rectangle.

4. The areas of all parallelograms, which have equal bases and heights, are equal to one another.

5. Parallelograms upon equal bases and between the same parallels are equal to one another.

6. The area of a parallelogram is found by multiplying the basis given in rods, feet inches, &c. by the height, expressed in units of the same kind.

7. The areas of parallelograms are to each other, as the products obtained by multiplying the length of the bases of the parallelograms by their heights.

8. Rectangles, or parallelograms which have equal bases, are to each other as their heights.

9. Rectangles, or parallelograms which have equal heights, are to each other as their bases.

10. If two triangles stand on the same basis and have equal heights, their areas are equal to one another.

11. Every triangle is half of a parallelogram upon an equal basis and of the same height.

12. The area of a triangle is half of the area of a parallelogram upon an equal basis and of the same height; and, therefore, the area of a triangle is found by multiplying the length of the basis by the height, and dividing the product by 2.

13. The areas of triangles upon the same basis and between the same parallels are equal.

14. The areas of triangles are to each other, as the products obtained by multiplying the length of their bases by their heights.

15. The areas of triangles upon equal bases are to each other, as the heights of the triangles.

16. The areas of triangles, which have equal heights, are to each other, as the bases of the triangles.

17. The area of a trapesoid is found by multiplying the sum of the two parallel sides by their distance.

18. The area of any rectilinear figure terminated by any number of sides, is found by dividing that figure, either by diagonals or by any other means, into triangles, and then adding the areas of these triangles.

19. If upon each of the three sides of a right-angular triangle, a square is constructed, the square upon the hypothenuse equals in area the two squares constructed upon the two sides, which include the right angle.

20. The areas of similar triangles are to each other, as the squares constructed upon the sides opposite to the equal angles, and also as the squares upon the heights of the triangles.

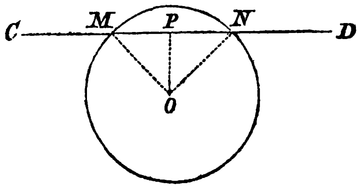
21. The areas of similar polygons are to each other, as the squares constructed upon the corresponding sides.

SECTION IV.

OF THE PROPERTIES OF THE CIRCLE.*

QUERY I.

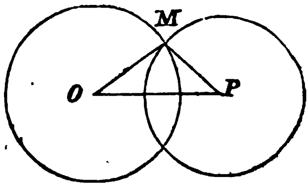
In how many points can a straight line CD, cut the circumference of a circle?



A. In two points M, N, only. For, dropping from the centre of the circle the perpendicular OP upon the straight line CD, there is but *one* point in the line CD, on each side of the perpendicular, such, that a line, drawn from it to the point O of the perpendicular, has the length of the radius ON. (page 63, 6*thly*.)

* Before entering on this section, the teacher ought to recapitulate with his pupils the definitions of a circle, of an arc, of a chord, a segment &c. (page 9.)

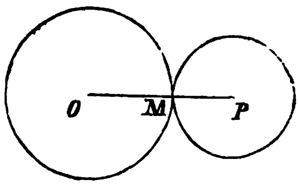
QUERY II.



In what cases do the circumferences of two circles cut each other?

A. When the distance OP between their centres, O and P, is less than the sum of their radii OM, PM.

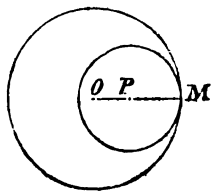
QUERY III.



When do two circles touch each other interiorly?

A. When the distance OP, between their centres, O and P, is equal to the sum of their radii OM, PM.

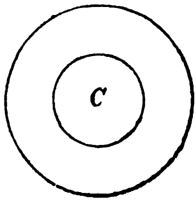
QUERY IV.



When do two circles touch each other interiorly?

A. When the distance OP between their centres, O and P, is equal to the difference between their radii, OM and PM.

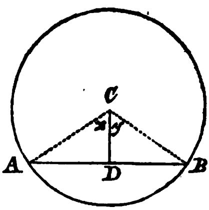
QUERY V.



When are the circumferences of two circles parallel to each other?

A. When the two circles are described from the same point C as a centre; that is, when they are concentric.

QUERY VI.



If from the centre C of a circle, a perpendicular CD is dropped upon a chord, AB, in that circle, what relation do the two parts AD, BD, into which the chord AB is divided, bear to each other?

A. The two parts AD, BD, are equal to each other; that is, the chord AB is bisected in the point D.

Q. How can you prove this?

A. By drawing the two radii AC, BC, the right-angular triangle ACD will have the hypothenuse AC and the side CD, equal to the hypothenuse BC and the side CD in the right-angular triangle BCD, each to each; therefore these two triangles must be equal; (page 55) and

12

the side AD, in the triangle ACD, is equal to the side BD in the *equal* triangle BCD.

Q. What other truths can you infer from the one you have just established?

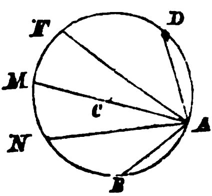
A. 1. *A straight line drawn from the centre of a circle to the middle of a chord, is perpendicular to that chord.*

2. *A perpendicular drawn through the middle of a chord, passes, when sufficiently far extended, through the centre of the circle.*

3. *Two perpendiculars, each drawn through the middle of a chord in the same circle, intersect each other at the centre; for each of them must go through the centre.*

4. *The two angles, x and y, which the radii AC, BC, drawn to the extremities of the chord AB, make with the perpendicular CD, are equal to one another;* for they are opposite to the equal sides AD, BD, in the equal triangles ADC, BDC.

QUERY VII.



If the two chords AD AB, are equal to each other, what remark can you make with regard to the arcs AD, AB, subtended by these chords?*

A. The two arcs AB, AD,

* The arcs AD, AB, standing on the chords AB, AD, are said to be *subtended* by these chords.

subtended by the equal chords AB, AD, are equal to one another.

Q. Why?

A. This follows from the perfect *uniformity* with which a circle is consructed. For if the chord AB is placed upon its equal the chord AD, the arcs, AB and AD, must coincide with each other; because every point in both these arcs is at the same distance from the centre C, of the circle.

Remark. It is to be observed that each chord subtends *two* arcs, one of which is *smaller* and the other *greater* than the semi-circumference, both together completing the *whole* circumference. In speaking of an arc, subtended by a chord, we always mean that which is *smaller* than the half-circumference.

Q. What other truths can you infer from the one you have just proved?

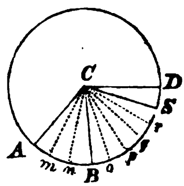
A. 1. *That equal arcs stand on equal chords;* for by placing one of the equal arcs AB, AD, upon the other, the *beginning* and *end* of the two chords AB, AD, and therefore the *whole* chords themselves coincide with each other.

2. *The greater arc stands on the greater chord, and the greater chord subtends the greater arc.* The chord AF, for instance, is greater than the chord AD; and the arc AF belonging to the greater chord AF, is also greater than the arc AD, belonging to the smaller chord AD.

3. ***Among all the chords AD, AF, AM, AN, AB, &c. which can be drawn in a circle, the diameter AM is the greatest;*** because the greatest arc, the semi-circumference, stands on it.

Remark. All that has been said of chords and arcs in the *same* circle, holds true also of chords and arcs in *equal* circles.

QUERY VIII.



What relation do you discover between the angles ACB, BCD, at the centre C of a circle, and the arcs AB, BD?

Ans. The angles ACB, BCD, at the centre of the circle, are to each other in the same ratio as the arcs AB, BD, intercepted by their legs.

Q. How can you show this?

A. I divide the whole of the arc AD successively into smaller and smaller parts, until one of the points of division shall have fallen upon B. Then, it is evident that by drawing to the points of division the radii C m, C n, C o, &c., the *angles* ACB and BCD are divided into as many equal parts as the *arcs* AB, BD, (for the sectors A m C, m n C, n BC, &c. will all coincide with each other, when they are placed upon one another;) and therefore the same ratio which exists between the arcs AB, BD,

exists also between the angles ACB, BCD. In our figure, we have the ratio of the arc AB to the arc BD as 3 to 6; and the same ratio (as 3 to 6) exists also between the angles ACB and BCD at the centre of the circle; that is, the arc BD is as many times greater than the arc AB, as the angle BCD is greater than the angle ACB, (Def. of Geom. Proportions).

What inference can you draw from the truth you have just advanced?

Ans. 1. *If the arcs AB, BD, are equal to one another, the angles ACB, BCD, at the centre are also equal to one another; for they are in the same ratio as the arcs AB, BD, (namely, in the ratio of equality.)*

2. *If the angles ACB, BCD at the centre are equal to one another, the arcs AB, BD are also equal to one another; because they are to each other in the same ratio as the angles at the centre.*

Remark 1. It has already been stated (note to page 5) that angles are measured by arcs of circles, described with any radius between their legs. The reason is now apparent; for the arcs intercepted between their legs are *in proportion* to the angles at the centre.

Remark 2. If the circumference of a circle is divided into 360 equal parts, called degrees; each decree again into 60 equal parts, called minutes; each minute again into 60 equal parts, called seconds, &c. (page 6); it is easy to perceive, that the magnitude of an angle does not depend upon the *greatness*

5. Parallelograms upon equal bases and between the same parallels are equal to one another.

6. The area of a parallelogram is found by multiplying the basis given in rods, feet inches, &c. by the height, expressed in units of the same kind.

7. The areas of parallelograms are to each other, as the products obtained by multiplying the length of the bases of the parallelograms by their heights.

8. Rectangles, or parallelograms which have equal bases, are to each other as their heights.

9. Rectangles, or parallelograms which have equal heights, are to each other as their bases.

10. If two triangles stand on the same basis and have equal heights, their areas are equal to one another.

11. Every triangle is half of a parallelogram upon an equal basis and of the same height.

12. The area of a triangle is half of the area of a parallelogram upon an equal basis and of the same height; and, therefore, the area of a triangle is found by multiplying the length of the basis by the height, and dividing the product by 2.

13. The areas of triangles upon the same basis and between the same parallels are equal.

14. The areas of triangles are to each other, as the products obtained by multiplying the length of their bases by their heights.

15. The areas of triangles upon equal bases are to each other, as the heights of the triangles.

16. The areas of triangles, which have equal heights, are to each other, as the bases of the triangles.

17. The area of a trapesoid is found by multiplying the sum of the two parallel sides by their distance.

18. The area of any rectilinear figure terminated by any number of sides, is found by dividing that figure, either by diagonals or by any other means, into triangles, and then adding the areas of these triangles.

19. If upon each of the three sides of a right-angular triangle, a square is constructed, the square upon the hypothenuse equals in area the two squares constructed upon the two sides, which include the right angle.

20. The areas of similar triangles are to each other, as the squares constructed upon the sides opposite to the equal angles, and also as the squares upon the heights of the triangles.

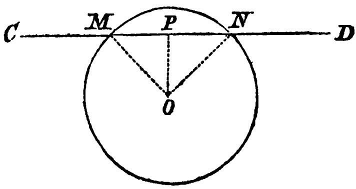
21. The areas of similar polygons are to each other, as the squares constructed upon the corresponding sides.

SECTION IV.

OF THE PROPERTIES OF THE CIRCLE.*

QUERY I.

In how many points can a straight line CD, cut the circumference of a circle?



A. In two points M, N, only. For, dropping from the centre of the circle the perpendicular OP upon the straight line CD, there is but *one* point in the line CD, on each side of the perpendicular, such, that a line, drawn from it to the point O of the perpendicular, has the length of the radius ON. (page 63, 6*thly*.)

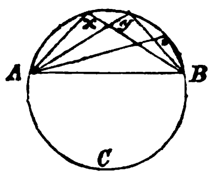
* Before entering on this section, the teacher ought to recapitulate with his pupils the definitions of a circle, of an arc, of a chord, a segment &c. (page 9.)

chord AB is measured by half the arc AB, as has been proved in the last query; and for the same reason is the angle z measured by half the arc BF; and therefore the remaining angle y is measured by half the arc AF; because half of the arc AF, together with half the arcs AB and BF, makes half the circumference. But the angle w at the centre, is measured by the *whole* arc AF; therefore the angle w is twice as great as the angle y.

Q. What important truths can you infer from the one you have just learned?

A. That every angle made by two chords at the circumference of a circle measures half as many degrees, minutes, seconds, &c. as the arc on the extremity of which those chords stand.

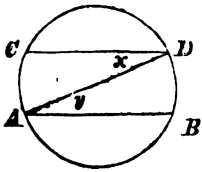
2. The angles x, y, z, at the circumference, having their legs standing on the extremities of the same arc ACB, are all equal to one another; because each of them is measured by half the arc ACB* (last query.)



* The arc ACB is designated by three letters, because some might understand the upper arc AB.

QUERY XII.

If two chords AB, CD in the same circle are parallel to each other, what relation do the arcs AC, BD, intercepted by them, on both sides of the circumference, bear to each other?



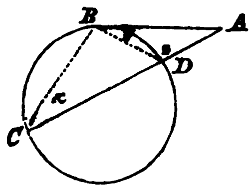
A. The arcs AC, BD, are equal to each other.

Q. How can you prove it?

A. Joining AD, the alternate angles x and y are equal to one another, (Query 11, Sect. I.) therefore the arcs AC and BD, by which these angles are measured, must also be equal to one another.

QUERY XIII.

Q. *If from the same point A, without a circle, you draw a tangent AB to the circle, and, at the same time, another line AC, cutting the circle; what relation exists between the tangent AB, and the line AC, which cuts the circle?*



A. The tangent AB is a mean proportional (see note to page 89) between the whole line AC and the part AD, which is without the circle.

Q. How can you prove it?

A. By joining BD and BC, the triangle ABD is similar to the whole triangle ABC; for the angle at A is common to both triangles, and the angle y in the triangle ABD is equal to the angle x in the triangle ABC (because both these angles are measured by half the arc BD*); and if two angles in one triangle are equal to two angles in another, each to each, the two triangles are similar to each other (page 87, 1st).

Q. But of what use is your proving that the triangle ABD is similar to the triangle ABC?

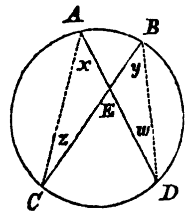
A. Because if the triangles, ABD and ABC, are similar to each other, the sides opposite to the equal angles in these triangles are in proportion, and therefore we have the proportion AD : AB = AB : AC, where the tangent AB is a mean proportional between the whole line AC and the part AD without the circle. (The sides, AD and AB, in the triangle ABD are opposite to the angles y and z in the same triangle, and the sides AB and AC in the triangle ABC are opposite to the angles x and CBA, which are respectively equal to the angles y and z).

* The angle x is formed at the circumference by the two chords BC and DC whose extremities stand on the arc BD; (Query 11, Sect. IV.) and the angle y is formed by the tangent BA, and the chord BD, which subtends the arc BD. (Query 10, Sect. IV.)

Q. Will you repeat the whole of your reasoning, and show that if from a point without a circle you draw a tangent, and a line cutting the circle, the tangent is a mean proportional between the line and the part of it without the circle?

* * *

QUERY XIV.



If two chords AD, BC, cut each other within the circle, what relation exists between the parts AE, ED, BE, EC, into which they mutually divide each other?

A. The two parts EC, EA, are in the inverse ratio of the two parts ED, EB; that is, we shall have the proportion

$$EC : EA = ED : EB.^*$$

Q. How can you prove it?

A. Joining AC and BD, the angle w is equal to the angle z; because each of these two angles w, z, measures half as many degrees as the arc AB; for the same reason is the angle x equal to the angle y; because each of these

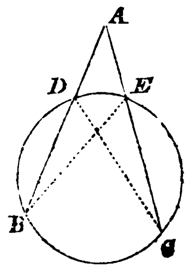
* The ratio ED to EB, is called inverse or inverted, because the two parts ED, EB, are not in *direct* proportion to the two parts EC, EE; that is, you cannot say the part EC of the chord BC is to the part EA of the chord AD, as the other part EB of the first chord BC is to the other part ED of the chord AD.

angles measures half as many degrees as the arc CD (Query 11, Sect. IV); and the angles AEC, BED, are also equal to each other, being opposite angles at the vertex (Query 5, Sect. I.); therefore the three angles of the triangle AEC are equal to the three angles of the triangle BED, each to each; consequently these two triangles are similar to one another; (page 87, 1st), and the sides opposite to the equal angles in both triangles are in a proportion. Thus we have

$$EC : EA = ED : EB;$$

(EC and EA are opposite to the angles x and z in the triangle AEC; and ED and EB, are opposite to the angles y and w, which are equal to the angles x and z, each to each.)

QUERY XV.



If from a point A, without a circle, two lines AB, AC, are drawn, cutting the circle; what relation exists between the lines AB, AC, and their parts AD, AE, without the circle?

A. The whole lines AB, AC, are to each other in the inverse ratio of their parts AD, AE, without the circle; that is, we have the proportion,

AB : AC = AE : AD (see the note to page 145).

Q. Why is this so?

A. If you join BE and DC, the two triangles ABE and ADC are similar to each other; because two angles in the one are equal to two angles in the other, each to each (page 87, 1st); the angle at A, namely, is common to both, and the angles at B and C are equal; because they have the same measure (half the arc DE); and as in similar triangles the sides opposite to the equal angles are in proportion, we have

$$AB : AE = AC : AD,^*$$

or, by changing the order of the mean terms, (principle 2d of proportion)

$$AB : AC = AE : AD,$$

as above.

Remark I. A *regular polygon* is a rectilinear figure which has all its angles and all its sides equal to one another.

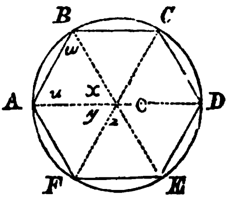
Remark II. A rectilinear figure is said to be *inscribed* in a circle, when the vertices of all the angles of that figure are at the circumference of the circle.

Remark III. A rectilinear figure is said to be *circumscribed* about a circle, when every side of that figure is a tangent to the circle.

---

* The teacher will do well to show his pupils again, that the sides, AB and AE, are the corresponding sides to AC and AD; because they are opposite to the equal angles in these triangles.

QUERY XVI.



If you divide the circumference of a circle into any number of equal parts, for instance into 6 parts, and then join the points of division by the chords AB, BC, CD, DE, EF, FA, what remark can you make respecting the rectilinear figure ABCDEF, which will be inscribed in the circle?

A. The figure thus inscribed in the circle is a regular polygon.

Q. How can you prove this?

A. The circumference of the circle being divided into equal parts, it follows that the arcs AB, BC, CD, &c. and consequently also the chords AB, BC, CD, &c. which form the sides of the inscribed figure, are equal to one another (page 135, 1st); and as each of the angles ABC, BCD, CDE, &c. has its legs standing on the whole circumference less two of the equal arcs, into which the circumference is divided, they all measure the same number of degrees, and consequently the angles of the inscribed figure are also equal to one another; *

* The angle ABC, for instance, has its legs standing on the whole circumference less the two arcs AB, BC; and the angle BCD has its legs standing on the whole circumference less the two equal arcs BC, CD, &c.

therefore the inscribed figure ABCDEF is a *regular* polygon.

Q. If in this manner you divide the circumference of a circle into 3, 4, 5, 6, &c. equal parts, what will be the magnitude of each of the arcs AB, BC, CD, &c.

A. Each of the arcs AB, BC, CD, &c. will then be $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, *&c. of the whole circumference, that is,* $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, *&c. of* 360 *degrees, according as the circumference has been divided into* 3, 4, 5, 6, *&c. parts.*

Q. And what do you observe with regard to the angles x, y, z, &c. at the centre of the circle, which the radii OA, OB, OC, &c. drawn to the points of division A, B, C, D, &c. make with each other?

A. That these angles x, y, z, &c. are all equal to one another; because they are measured by the equal arcs AB, BC, CD, &c. They will therefore measure $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. of 360 degrees, according as the circumference of the circle is divided into 3, 4, 5, &c. equal parts.

QUERY XVII.

Can you find the relation which one of the sides of a regular hexagon inscribed in a circle, bears to the radius of that circle? (See the figure belonging to the last Query.)

A. The side of a regular hexagon inscribed in a circle is equal to the radius of that circle.

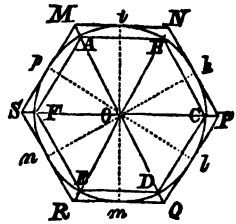
Q. Why?

A. Because each of the triangles ABO, BCO, CDO, &c., is in the first place isosceles, two of

its sides, being radii of the same circle; and as each of the angles x, y, z, &c. at the centre of the circle measures $\frac{1}{6}$ of 360, that is, 60 degrees (last Query); it follows that the two angles at the basis of each of the isosceles triangles ABO, BCO, CDO, &c., for instance the two angles w, and u, at the basis of the isosceles triangle ABO, measure together 120 degrees; because the sum of the three angles in a triangle is equal to two right angles, (Query 14, Sect. I,) and two right angles have for their measure half of the circumference or 180 degrees (page 138, Remark 3d), and 60 and 120 make 180 degrees; now as the two angles at the basis of every isosceles triangle are equal to each other (Query 3, Sect. II); each of the two angles at the basis of one of the isosceles triangles ABO, BCO, CDO, &c. will measure half of 120, that is, 60 degrees; therefore the three angles in each of the triangles ABO, BCO, CDO, &c. are equal to one another;* and therefore these triangles are not only isosceles, but also *equilateral* (page 48); therefore each of the sides AB, BC, CD of the hexagon is equal to the radius of the circle, (AB, OB, OC, &c.)

* For each of the angles at the centre measures also 60 degrees.

QUERY XVIII.



If in a regular polygon inscribed in a circle, you draw from the centre of the circle the radii O i, O k, O l, O m, &c. perpendicular to the chords AB, BC, CD, &c.; and at the extremities of these radii the tangents MN, NP, PQ, &c.; what do you observe with regard to the figure MNPQRS circumscribed about the circle?

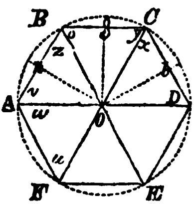
A. The figure MNPQRS, circumscribed about the circle, is a regular polygon, of the same number of sides as the inscribed polygon ABCDEF.

Q. How do you prove this?

A. The chords AB, BC, CD, &c. are perpendicular to the same radii, to which the tangents MN, NP, PQ, &c. are perpendicular; consequently the chords AB, BC, CD, &c. are parallel to the tangents MN, NP, PQ, &c. (for two straight lines, which are both perpendicular to a third line, are parallel to each other (Query 7, Sect. I.) and therefore the triangles ABO, BCO, CDO, &c. are all similar to the triangles MNO, NPO, PQO, &c. from which they may be considered as cut off by the lines AB, BC, CD, &c. being drawn parallel to the sides MN, NP, PQ, &c. (Query 16, Sect. II.) Now as the triangles ABO, BCO, CDO, &c. are all

equal to one another, the triangles MNO, NPO, PQO, &c. are all equal to one another. And therefore the circumscribed figure, MNPQRS, is a regular polygon, similar to the one inscribed in the circle.

QUERY XIX.



It has been proved (Query 16, *Sect. IV,) that a regular polygon, of any number of sides, may be* inscribed *in a circle, by dividing the circumference of the circle into as many equal parts, as the polygon shall have sides, and then joining the points of division by straight lines: can you now prove the reverse, that is, that around every regular polygon, a circle can be drawn in such a manner, that all the vertices of the polygon shall be at the circumference of the circle?*

A. Yes, for I need only bisect two adjacent sides of a regular polygon; for instance, the two sides AB, BC, of the regular polygon ABCDEF, and in the points of bisection, erect the two perpendiculars, *g* O, *k* O, which will necessarily cut each other in a point which I call O. Then it is evident, that by drawing the lines OB, OC, OA, these three lines are equal to each other; for the line OB is equal to OC because the two points B and C are at an equal distance from the perpendicular *g* O (page 53, 5thly); and

for the same reason is OB also equal to OA; because the points B and A are at an equal distance from the perpendicular k O. Thus we have in the two triangles ABO, BCO, the three sides in the one, equal to the three sides in the other, each to each; namely

the side AB = the side BC
" " OB = " " OB
and the side OB = the side OA = OC;

therefore these two triangles are both isosceles and equal to each other.

Q. But of what use is your proving that the triangle ABO is equal to the triangle BCO?

A. It shows that each of the angles in the polygon is bisected by one of the lines OA, OB, OC. For in the first place, we have in the two equal triangles BCO, ABO, the angle o equal to the angle z (these two angles being opposite to the equal sides OC, OA); therefore the angle ABC is bisected; and the angle o is further equal to the angle y (because the triangle BCO is isosceles; (Query 3, Sect. II); and as the angle o is half of the angle ABC, the angle y, its equal, is also half of the angle BCD; which in the regular polygon ABCDEF must be equal to the angle ABC; therefore the angle BCD is also bisected, and for the same reason is the angle v half of the angle FAB, and therefore the angle FAB bisected. And now I can show, that

drawing from the point O the lines OF, OE, OD, to the remaining vertices F, E, D, the whole polygon is divided into equal isosceles triangles. Taking in the first place the two triangles, AFO and ABO; they have two sides OA, FA, in the one, equal two sides, OA, AB, in the other, each to each; and as the angle FAB is bisected by the line OA, the two angles *v* and *w* are also equal; therefore the two triangles AFO, ABO are equal to each other (Query 1, Sect. II); and as the triangle ABO is isosceles, the triangle AFO is also isosceles; therefore the angle *u* is equal to the angle *w* (Query 3, Sect. II); and as the angle *w* is half of the polygon angle FAB, the angle *u* is also half of the equal polygon angle AFE. In precisely the same manner it may be proved that the triangles FEO, is isosceles and equal to the triangle AFO; and then that the triangle EDO is isosceles and equal to the triangle FEO; and as the whole polygon ABCDEF is thus divided into equal isosceles triangles, the lines OA, OB, OC, OD, OE, OF, are all equal to one another; and therefore by describing from the point O, as a centre, with a radius OA, a circle around the polygon ABCDEF, each of the vertices A, B, C, D, E, F, will be a point in the circumference of the circle.

Q. What other important consequence follows from the principle you have just proved?

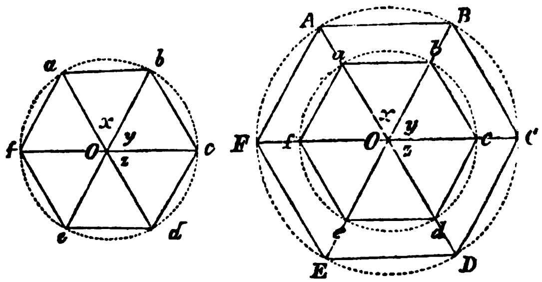
A. 1. *That in every regular polygon a circle may be inscribed in such a manner, that every side of the polygon is a tangent to the circle.* For if in the regular polygon ABCDEF, you describe with a radius *O g* the circumference of a circle, that circumference will touch the middle of the sides AB, BC, CD, DE, EF, FA, of the polygon ABCDEF; because the lines O *k*, O *g*, O *l*, &c. are all equal to one another, and will therefore be radii of the inscribed circle; and the sides AB, BC, CD, &c. being perpendicular to the radii O *k*, O *g*, O *l*, &c. will all be tangents to that circle (page 138.)

Q. Can you now prove that the same principles you have demonstrated with regard to a regular hexagon, hold true also with regard to a regular polygon of 7, of 8, of 9, or any other number of sides?

*

* *

QUERY XX.



What relation do you observe to exist between two regular polygons a b c d e f, ABCDEF of the same number of sides?

A. They are similar to one another?

Q. How can you prove it?

A. By describing a circle around each of the regular polygons *a b c d e f*, ABCDEF, and drawing the radii O *a*, O *b*, O *c*, O *d*, O *e*, O *f*, OA, OB, OC, OD, OE, OF, each of these polygons is divided into as many equal triangles, as there are sides in the polygon (last Query); and as all the angles *x*, *y*, *z*, &c. formed at the centre of the regular polygon, inscribed in a circle, are equal to one another (for they are all measured by equal arcs of the circumference of the circumscribed circle), I can place the centre O, of the polygon *a b c d e f*, upon the centre O of the polygon ABCDEF, in such a manner, that

the angles at the centre shall all coincide with each other; namely so, that the radius O *a*, shall fall upon the radius OA, O *b* upon OB, O *c* upon OC, &c. Then, it is evident, that the sides *a b*, *b c*, *c d*, *d e*, &c. of the smaller polygon *a b c d e f*, are *parallel* to the sides AB, BC, CD, DE, &c. of the greater polygon ABCDEF; for the points *a*, *b*, *c*, *d*, *e*, *f*, and A, B, C, D, E, F, are in the circumferences of concentric circles (Query 5, Sect. IV); therefore the triangles *O a b*, *O b c*, *O c d*, &c. in the smaller polygon, are all similar to the triangles OAB, OBC, OCD, &c. in the greater polygon (Query 16, Sect. II.); consequently the whole polygon *a b c d e f* is similar to the whole polygon ABCDEF.

Q. What other truths can you infer from the one you have just learned?

A. 1. *The sums of all the sides of two regular polygons of the same number of sides are to each other in the same ratio, as the radii of the inscribed or circumscribed circles.* For in the two triangles, ABO and *a b o*, for instance, we have the proportion

$$AB : a\,b = AO : a\,o; \text{ that is,}$$

the side AB is as many times greater than the side *a b*, as the radius AO is greater than the radius *a o*; and therefore 6 or any other number of times the side AB, is as many times greater

than the same number of times the side *a b*, as the radius OA is greater than the radius *o a*; that is, the sum of all the sides of the regular polygon ABCDEF, is as many times greater than the sum of all the sides of the regular polygon *a b c d e f*, as the radius OA of the circle, circumscribed about the regular polygon ABCDEF, is greater than the radius *o a* of the circle, circumscribed about the regular polygon *a b c d e f*; and in the same manner I can prove, that the sum of all the sides of the regular polygon ABCDEF is as many times greater than the sum of all the sides of the regular polygon *a b c d e f*, as the radius of the circle, *inscribed* in the regular polygon ABCDEF, is greater than the radius of the circle inscribed in the regular polygon *a b c d e f*.

Remark. The sum of all the sides of a geometrical figure, that is, a line as long as all its sides together, is called the *perimeter* of that figure. The above proportion may therefore be expressed in shorter terms; namely, the perimeters of two regular polygons of the same number of sides, are to each other in the proportion of the radii of the inscribed or circumscribed circles.

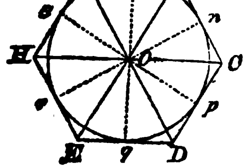
2. *The areas of two regular polygons of the same number of sides are in the same ratio, as the squares constructed upon the radii of the inscribed or circumscribed circles. Thus the area of the regular polygon ABCDEF is as many times greater than the area of the regular polygon a b c d e f, as the area of the square upon the radius OA is greater than the area of the*

square upon the radius o a. For the areas of the similar triangles ABO, *a b o* are to each other, as the squares upon the corresponding sides, the radii OA, *o a;* and therefore any number of times (in our figure 6 times) the areas of these triangles, that is, the areas of the regular polygons ABCDEF, *a b c d e f,* are to each other in the same ratio; and in the same manner I can prove, that the area of the polygon ABCDEF is as many times greater than the area of the polygon *a b c d e f*, as the square upon the circle *inscribed* in the regular polygon ABCDEF, is greater than the square upon the radius of the circle inscribed in the regular polygon *a b c d e f.*

Q. Will you now prove that the same principles hold true with regard to two regular polygons of 7, of 8, of 9, &c. sides?

* * *

QUERY XXI.



From what you have learned of the properties of regular polygons, can you give a rule for finding the area of a regular polygon?

A. Yes. Multiply the sum of all the sides, (the

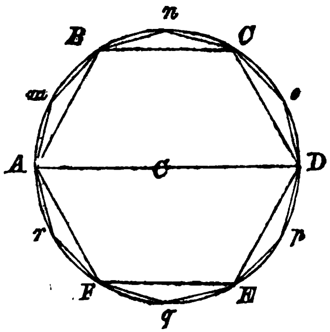
perimeter) of the regular polygon, by the radius of the inscribed circle; the product divided by 2, will be the area of the regular polygon.

Q. Why?

A. Because every regular polygon, the polygon ABCDEF, for instance, can be divided into as many equal triangles, as there are sides in the polygon; and the area of each of these triangles is found by multiplying the basis, that is, one of the sides of the polygon ABCDEF, by the height (which, in every one of these triangles, is equal to the radius *o m* of the inscribed circle) and dividing the product by 2; therefore the area of the whole polygon ABCDEF may at once be found by multiplying the *sum* of all the sides by the radius of the inscribed circle, and dividing the product by 2.*

* Instead of multiplying the perimeter by the whole radius, and then dividing the product by 2, you may at once multiply the perimeter by *half* the radius, or the radius by half the perimeter.

QUERY XXII.



If you bisect each of the arcs AB, BC, CD, &c. subtended by the sides AB, BC, CD, &c. of a regular polygon inscribed in a circle; and then to the points of division m, n, o, p, q, r, draw the lines Am, mB, Bn, n C, C n, &c. what do you observe with regard to the regular polygon Am B n C o D p E q F r thus inscribed in the circle?

A. The regular polygon Am B n C o D p &c. has twice as many sides, as the regular polygon ABCDEF; for the circumference of the circle is now divided into twice as many equal parts as before. Thus if the regular polygon ABCDEF has 6 sides, the regular polygon A*m*B*n*C*o*D*p* &c. has 12 sides; and by bisecting again the arcs A *m*, *m* B, B *n*, &c. I can inscribe a regular polygon of 24 sides, and so

on, by continuing to bisect the arcs, a regular polygon of 48, 96, 192, &c. sides.

Q. And what do you observe with regard to the arcs, which are subtended by the sides of the polygons, ABCDEF and A *m* B *n* C *o* D *p* E *q* F *r*, inscribed in the circle?

A. The arcs AB, BC, CD, &c. subtended by the sides of the regular polygon ABCDEF first inscribed in the circle, stand further off the sides AB, BC, CD, &c. than the arcs A *m*, *m* B, B *n*, &c. stand off the sides A *m*, *m* B, B *n*, &c. of the regular polygon of twice the number of sides; and therefore, were the arcs AB, BC, CD, &c. drawn out into straight lines, they would differ more from the sides AB, BC, CD, &c. of the regular polygon ABCDEF, first inscribed in the circle, than the arcs A *m*, *m* B, B *n*, *n* C, &c. drawn out into straight lines, would differ from the sides A *m*, *m* B, B *n*, &c. of the regular polygon A *m* B *n* C *o* &c. of twice the number of sides.

Q. Now, if continuing to bisect the arcs, you inscribe regular polygons of 24, 48, 96, 192, &c. sides, what further remark can you make with regard to the arcs, subtended by the sides of these polygons?

A. These arcs differ less in length from the sides which subtend them, in proportion as the polygon consists of a greater number of sides; because by continuing to bisect the

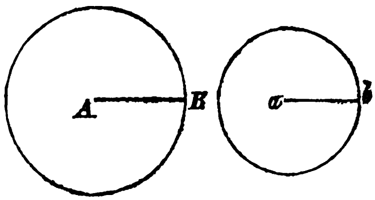
arcs, and thereby to increase the number of sides of the inscribed polygons, the arcs subtended by these sides become nearer and nearer to the sides themselves, and finally the difference between them will become imperceptible.

Q. And what conclusion can you now draw respecting the whole circumference of a circle?

A. That the circumference of a circle differs very little from the sum of all the sides of a regular inscribed polygon of a great number of sides; therefore if the number of sides of the inscribed polygon is very great, (several thousand for instance) the polygon will differ so little from the circle itself, that without perceptible error the one may be taken for the other.

QUERY XXIII.

It has been shown in the last query, that a circle may be considered as a regular polygon of a very great number of sides; what inferences can you now draw with regard to the circumferences and areas of circles?



A. 1. ***The circumferences of two circles are in proportion to the radii of these circles; that is, a straight line as long as the circumference of the first circle, is as many times greater than***

a straight line, as long as the circumference of the second circle; as the radius AB of the first circle, is greater than the radius a b of the second circle. For if in each of the two circles a regular polygon of a very great number of sides is inscribed, the sums of all the sides of the two polygons are to each other in proportion to the radii, AB, *a b*, of the circumscribed circles (page 157, 1st); and as the difference between the circumference of a circle and the sum of all the sides of an inscribed polygon of a great number of sides is imperceptible (last Query), we may say that the circumferences themselves are in the same ratio as the radii AB, *a b*.*

2. *The areas of two circles are in proportion to the squares constructed upon their radii; that is, the area of the greater circle is as many times greater than the area of the smaller circle, as the area of the square upon the radius of the greater circle, is greater than the area of the square constructed upon the radius of the smaller circle.* For if in each of these circles a regular polygon of a great

* The teacher may give an ocular demonstration of this principle by taking two circles, cut out of pasteboard or wood; and measuring their circumferences by passing a string around them. The measure of the one will be as many times greater than the measure of the other, as the radius of the first circle is greater than the radius of the second circle.

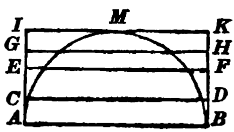
number of sides is inscribed, the difference between the areas of the polygons and the areas of the circles themselves will be imperceptible; and because the areas of two regular polygons of the same number of sides are in the same ratio, as the areas of the squares upon the radii of the circles in which they are inscribed, (page 158, 2dly) the areas of these circles will themselves be in the ratio of the squares upon their radii.

3. *The area of a circle is found by multiplying the circumference of the circle, given in rods, feet, inches, &c. by half the radius, given in units of the same kind.* Because a circle differs so little from a regular inscribed polygon of a great number of sides, that the area of the polygon may, without perceptible error, be taken for the area of the circle. Now, the area of regular polygon inscribed in a circle, is found by multiplying the sum of all the sides by the radius of the *inscribed* circle (page 159) and dividing the product by 2; therefore the area of the circle itself is found by multiplying the circumference (instead of the sums of all the sides of the inscribed polygon) by the radius, and dividing the product by 2. For it has been shown in the last Query, that the sides of a regular inscribed polygon grow nearer and nearer the circumference of the circumscribed circle, in proportion

as these sides increase in number; consequently the circumference of a circle, *inscribed* in a regular polygon of a great number of sides, will also grow nearer and nearer the circumference of the circumscribed circle; until finally the two circumferences will differ so little from each other, that the radius of the one may, without perceptible error, be taken for the radius of the other.

Remark. Finding the *area of a circle* is sometimes called *squaring the circle*. The problem, to construct a *rectilinear figure*, for instance a rectangle, whose area shall exactly equal the area of a given circle, is that which is meant by finding the *quadrature of the circle*. For the area of any geometrical figure, terminated by straight lines only, can easily be found by the rule given in Query 5, Sect. III; or, in other words, we can always construct a square, which shall measure exactly as many square rods, feet, inches, &c. as a rectilinear figure of any number of sides.

Now it is easy to show, that there is nothing absurd in the idea of constructing a rectilinear figure, for instance a rectangle, whose area shall be equal to the area of a given circle. For let us take a semicircle ABM, and let us for a moment imagine the diameter AB to move parallel to itself between the two perpendiculars AI, BK. It is evident that when the diameter AB is very near its original position, for instance in CD, the area of the rectangle ABCD is *smaller* than the area of the semicircle ABM; but the diameter continuing to move parallel to itself in the direction from A to I, there will be a point in the line AI, where the area of the rectangle ABIK is *greater* than the area of the semicircle ABM. Now as there is a point in the line AI, below the point I, in which the area of the rectangle ABCD is smaller than the area of the



semicircle ABM, and as the diameter by continuing to move in the same direction makes in different points C, E, G, &c. of that same line, the rectangles ABCD, ABEF, ABGH, &c. whose areas become greater and greater, until finally they become greater than the area of the semicircle itself; there must evidently be a point in the line AI, in which a line drawn parallel to the diameter AB makes with it and the perpendiculars AI, BK, a rectangle, which, in area, is *equal* to the semicircle ABM; and as there is a rectangle which, in area, is equal to the semicircle ABM, by doubling it, we shall have a rectangle which, in area, is equal to the whole circle.

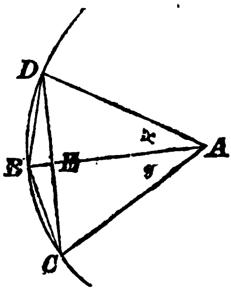
Neither is it difficult to find the area of a circle *mechanically*. For the area of a circle being found by multiplying the circumference by the length of the radius, and dividing the product by 2 (page 165, 3dly), we need only pass a string around the circumference of a circle, and then multiply the length of that string by the length of the radius; the product divided by 2 will be the area of the circle. Having thus found the comparative length of the radius and circumference of one circle, we might determine the circumference, and thereby the area of any other circle, when knowing its radius. For the circumferences of two circles being in proportion to the radii of the two circles, we should have three terms of a geometrical proportion given; *viz.* the radii of the two circles, and the circumference of the one; from which we might easily find the fourth term (theory of Proportions, page 75, 6thly), which would be the circumference of the other circle.

But the expressions of the circumference and area of a circle, thus obtained by measurement, are never so correct as is required for very nice and accurate mathematical calculations; we must therefore resort to other means such as geometry itself furnishes, to *calculate* the ratio of the radius or diameter to the circumference of the circle; and herein consists the difficulty of the quadrature of the circle. For if the ratio of the radius to the circumference is once determined, we can easily find the circumference of any circle, when its radius is given; and knowing the circumference and the radius, we can find the area of the circle. (page 165, 3dly).

To calculate the ratio of the *diameter* to the circumference, mathematicians have compared the circumference of a circle to the sum of all the sides of a regular inscribed polygon of a great number of sides; for it has been shown (page 163) that the circumference of a circle differs very little from the sum of all the sides of such a polygon.

For this purpose they took a regular inscribed *hexagon*, each of the sides of which is equal to the radius of the circumscribed circle (Query 17, Sect. IV.) For the sake of convenience they supposed the diameter of the circle equal to unity; the radius, and therefore the side of a regular inscribed hexagon is then ½, and the sum of all the sides (6 times ½) equal to 3. This is the first approximation to the circumference of the circle.

From the side of a regular inscribed hexagon, it is easy to find that of a regular inscribed polygon of 12 sides. Supposing, for instance, the chord CD to be the side of a regular inscribed hexagon, by bisecting the arc CD in B, the chords BC, BD, will be two sides of a regular inscribed polygon of 12 sides, the length of which can easily be calculated when the chord CD and the radius AC are once known. For the radius AB *bisects* the chord CD; because the two triangles ACE and ADE, have the two sides AE, AC, in the one, equal to the two sides AE, AD in the other, each to each (AC and AD being radii of the same circle), and have the included angles, x and y, also equal (these two angles being measured by the equal arcs BC, BD); therefore the two triangles ACE and ADE are equal to one another; and the side EC in the one is equal to the side ED in the other. But the radius AB which bisects the chord CD is perpendicular to that chord (see page 134, 1st.); therefore the two triangles ACE, ADE, are both right-angular, and the radius AC and EC (half of CD) being known, the hypothenuse (AC) and one of the sides (the side EC) of the right-angular triangle AEC are given; whence it is easy to find the other side AE, by the rule given in



the remark, page 121. Thus if the radius is supposed to be $\frac{1}{2}$, the side CD of the inscribed hexagon is also equal to $\frac{1}{2}$; and EC (half of CD) is $\frac{1}{4}$. Taking the square of $\frac{1}{4}$ from that of $\frac{1}{2}$, and extracting the square root of the remainder, we obtain the length of the side AE, which, subtracted from the radius AB, leaves the length of BC. Now we can find the side BC, in the rightangular triangle BCE, by extracting the square root of the sum of the squares of BC and EC (see the remark, page 121); and one of the sides of the regular inscribed polygon of 12 sides being once determined, we need only multiply it by 12, in order to obtain the sum of all its sides, which is the second approximation to the circumference of the circle. In precisely the same manner can the sides, and consequently also the *sum* of all the sides of a regular inscribed polygon of 24 sides be obtained; which is the third approximation to the circumference. Thus we might go on finding the sum of all the sides of a regular inscribed polygon of 48, 96, 192, &c. sides, until the inscribed polygon should consist of several thousand sides: the sum of all the sides would then differ so little from the circumference of the circle, that, without perceptible error, we might take the one for the other.

In this manner the approximation to the circumference of the circle has been carried further than is ever required in the minutest and most accurate mathematical calculations.

The beginning of this extremely tedious calculation gives the following results.

Parts of the circumference.	Side of the inscribed polygon.	Sum of all the sides of the inscribed polygon.
6	0,5	3
12	0,258819	3,105828
24	0,130526	3,132628
48	0,065403	3,139348
96	0,032719	3,141033
192	0,016361	3,141446

It is not necessary to carry this calculation any further, since analysis furnishes us with means to obtain the same results in a much easier manner.

In nearly the same manner LUDOLPH VAN CEULEN first found the ratio of the diameter to the circumference of a circle to 32 decimals. (See his 'Arithmetische en Geom. Fundamenten, page 163. Leiden. 1616; also his work 'De Circulo et Adscriptis, c. 10. Leiden. 1619.'*)

ARCHIMEDES found the ratio of the diameter to the circumference as near as 7 to 22.

FRANCISCUS VIETA found it as 1 to 3,1415926535.

ADRIANUS ROMANUS added the following decimals

89793.

LUDOLPH VAN CEULEN added further

23846264338327950288.

SHARP added again

41971693993751058209749445923078,

To which MACHIN further added

164062862089986280348253421170679,

And lastly LAGNY increased them by

8214808651328230664709384446.

In a manuscript in the library at Oxford this number is still farther extended by 29 decimals, namely,

46095505182231725359408128480 2.

So that the most accurate ratio of the diameter to the circumference is at present as

1 to 3,1415926535897932384626433832795028841971693993751
058209749445923078164062862089986280348253421170 6
798214808651328230664709384446460955051822317253 59
4081284802.

The last ratio is so near the truth, that in a circle, whose diameter is one hundred million times greater than that of the

---

* The first work which LUDOLPH VAN CEULEN published on this subject, bears the title 'Van den Circkel, daer in gheleert wird te vinden de naeste proportie des Cirkels-Diameter tegen synen Omloop. Leiden. d. 20 Sept. 1596.' The work is dedicated to Prince MORIZ VON ORANIEN. The titles of these books are mentioned here, merely because MONTUCLA, in his 'Histoire de la Quadr. du Circle,' page 49, in speaking of LUDOLPH VAN CEULEN, says there is hardly anything known of him; which he could not have said, if he had only read the title pages of his works.

sun, the error would not amount to the one hundred millionth part of the breadth of a hair.

In general, when the calculations need not be very minute and accurate, 7 decimals will suffice. Thus we may consider the diameter to the circumference to be

as 1 to 3,1415926; that is,

if the diameter of a circle is 1, its circumference is 3,1415926;* consequently if the diameter is 2, or the radius 1, the circumference will be twice 3,1415926, equal to 6,2831852.

Dividing this number by 360 we obtain the length of a degree; dividing the length of a degree by 60, we obtain the length of a minute; and that again divided by 60, gives the length of a second, and so on. In this manner we obtain the length of

1 degree	equal	to	0,0174533†
1 minute	"	"	0,0002909
1 second	"	"	0,0000048
1 third	"	"	0,0000001

Having once determined the circumference of the circle whose radius is 1, we can easily find the circumference of any circle, when its radius is given; for we need only multiply the number 6,2831352 (that is the circumference of a circle whose radius is 1) by the radius of the circle whose circumference is to be found, the product will be the circumference sought. Thus if it be required to find the circumference of a circle whose radius is 6 inches, we need only multiply the number 6,2831852 by 6; the product 37,6991112 is the circumference of that circle.

If it be required to find the length of an arc of a given number of degrees, minutes, seconds, &c. in a given circle, we need only multiply

the degrees by 0,0174533
the minutes by 0,0002909
the seconds by 0,0000048, &c.;

the different products added together give the length of an arc of the same number of degrees, minutes, seconds, &c. in the circle

---

* The number 3,1415926 is sometimes represented by the Greek letter π. Thus the circumference of a circle, whose radius is 1, may be represented by 2π.

† The last figure in these expressions has been corrected.

whose radius is 1; and multiplying this product by the radius of the given circle, we shall have the length of the arc sought. If, for instance, it be required to find the length of an arc of 5 degrees and 2 minutes, in a circle whose radius is 6 inches, we in the first place multiply 0,0174533 by 6, and

0,0002909 by 2; the products of

these multiplications, 0,1047198 } added together
and 0,0005818 }

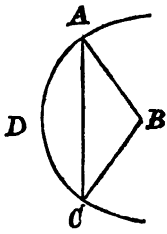
give 0,1053016, which is the length of an arc of 5 degrees and 2 minutes in the circle whose radius is 1, and this last product (0,1053016), multiplied by 6, gives 0,6318096, which is the length of an arc of the same number of degrees and minutes of a circle whose radius is 6 inches.

Now that we are able to find the circumference, or an arc of the circumference of any circle, when knowing its radius, nothing can be easier than to calculate the area of a circle, of a sector, a segment, &c.

The area of a circle being found by multiplying the circumference by half the radius; or by multiplying half the circumference by the whole radius, (page 165), we need only take the number 3,1415926, which is half the circumference of the circle whose radius is 1, and multiply it by the radius, of the given circle, the product will be half the circumference of the given circle, which multiplied again by the radius, gives us the area of it. Thus if it is required to find the area of a circle whose radius is 5 inches, we multiply the number 3,1415926 twice in succession by 5, that is, we multiply it by the *square* of 5*; the product 78,5398150 is the area sought. Hence follows the general rule: in order to find the area of a circle multiply the number 3,1415926 by the square of the radius. If the radius is given in rods, the answer will be square rods; if given in feet, the answer will be square feet; if in seconds, square seconds, and so on. The area of a semicircle is found by dividing the area of the whole circle by 2. In the same manner we find the area of a quadrant by dividing the area of the whole circle by 4, &c.

---

* Multiplying a number twice in succession by 5 is the same as multiplying that number by 25; which is the square of 5, because 5 times 5 are 25.



The area of a sector BCAD is found by multiplying the length of the arc CAD (see the rule, page 171) by half the radius BD; or we may first find what part of the circumference the arc CDA is; whether a third, a fourth, a fifth, &c. and then divide the area of the whole circle whose radius is BC, by 3, 4, 5, &c. according as the arc CDA is $\frac{1}{3}$, $\frac{1}{4}$, &c. of the whole circumference. If we are to find the area of the segment CDA, we must first find the area of the sector BCDA; and then also the area of the triangle ABC; which, subtracted from the area of the sector BCDA, will leave the area of the segment CDA.

RECAPITULATION OF THE TRUTHS CONTAINED IN FOURTH SECTION.

Q. Can you now repeat the different relations, which exist between the different parts of a circle and the straight lines, which cut or touch the circumference?

A. 1. A straight line can touch the circumference in only one point.

2. When the distance between the centres of two circles is less than the sum of their radii, the two circles cut each other.

3. When the distance between the centres of two circles is equal to the sum of their radii, the two circles *touch* each other *exteriorly*.

4. When the distance between the centres of two circles is equal to the difference between their radii, the two circles *touch* each other *interiorly*.

5. When two circles are concentric, that is, when they are both described from the same point as a centre, the circumferences of the two circles are parallel to each other.

6. A perpendicular, dropped from the centre of a circle upon one of the chords in that circle, divides that chord into two equal parts.

7. A straight line, drawn from the centre of a circle to the middle of a chord, is perpendicular to that chord.

8. A perpendicular drawn through the middle of a chord, passes, when sufficiently far extended, through the centre of the circle.

9. Two perpendiculars, each drawn through the middle of a chord in the same circle, intersect each other at the centre.

10. The two angles, which two radii drawn to the extremities of a chord, make with the perpendicular dropped from the centre of the circle to that chord, are equal to one another.

11. If two chords, in the same circle, or in equal circles, are equal to one another, the arcs subtended by them are also equal; and the reverse is also true; that is, if the arcs are equal to one another, the chords which subtend them are also equal.

12. The greater arc stands on the greater chord, and the greater chord subtends the greater arc.

13. The angles at the centre of a circle are to each other in the same ratio, as the arcs of the circumference intercepted by their legs.

14. If two angles at the centre of a circle are equal to one another, the arcs of the circumference intercepted by their legs are also equal; and the reverse is also true, that is, if the two arcs intercepted by the legs of two angles at the centre of a circle, are equal to one another, these angles are also equal.

15. Angles are measured by arcs of circles, described with any radius between their legs. The circumference is for this purpose divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes; each minute again into 60 equal parts, called seconds, &c.

16. The magnitude of an angle does not depend on the greatness of the arc intercepted by its legs; but merely on the number of degrees, minutes, seconds, &c. it measures of the circumference.

17. The circumference of a circle is the measure of 4 right angles; the semi-circumference that of 2 right angles; and a quadrant that of 1 right angle.

18. A straight line, drawn at the extremity of the diameter or radius perpendicular to it, touches the circumference only in one point, and is therefore a tangent to the circle.

19. A radius or diameter, drawn to the point of tangent, is perpendicular to the tangent.

20. A line, drawn through the point of a tangent perpendicular to the tangent, passes, when sufficiently far extended, through the centre of the circle.

21. The angle, formed by a tangent and a chord, is half of the angle at the centre, which is measured by the arc subtended by that chord; therefore the angle, formed by the tangent and the chord, measures half as many degreés, minutes, seconds, &c. as the angle at the centre.

22. The angle which two chords make at the circumference of a circle, is half of the angle made by two radii at the centre, if the legs of both these angles stand on the extremities of the same arc; therefore every angle, made by two chords, at the circumference of a circle, measures half as many degrees, minutes, seconds, &c. as the arc intercepted by its legs.

23. If several angles at the circumference have their legs standing on the extremities of the same arc, these angles are all equal to one another.

24. Parallel chords intercept equal arcs of the circumference.

25. If from a point without the circle you draw a tangent to the circle, and, at the same time, a straight line cutting the circle, the tangent

is a mean proportional between that whole line, and the part of it which is without the circle.

26. If a chord cuts another *within* the circle, the two parts, into which the one is divided, are in the inverse ratio of the two parts, into which the other is divided.

27. If from a point without a circle, two straight sides are drawn, cutting the circle, these lines are to each other in the inverse ratio of their parts without the circle.

28. If the circumference of a circle is divided into 3, 4, 5, &c. equal parts, and then the points of division are joined by straight lines, the rectilinear figure, thus inscribed in the circle, is a regular polygon of the same number of sides, as there are parts into which the circumference is divided.

29. If from the centre of a regular polygon inscribed in a circle, radii are drawn to all the vertices at the circumference, the angles which these radii make with each other at the centre, are all equal to one another.

30. The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.

31. If from the centre of a circle radii are drawn, bisecting the sides of a regular inscribed polygon, and then at the extremities of these radii tangents are drawn to the circle, these tangents form with each other a regular *circumscribed* polygon of the same number of sides as the regular inscribed polygon.

32. Around every regular polygon a circle can be drawn in such a manner, that all the vertices of the polygon shall be at the circumference of the circle.

33. Two regular polygons of the same number of sides are similar figures.

34. The sums of all the sides of two regular polygons of the same number of sides, are to each other in the same ratio, as the radii of the inscribed or circumscribed circles.

35. The areas of two regular polygons of the same number of sides, are to each other as the areas of the squares, constructed upon the radii of the inscribed or circumscribed circles.

36. The area of a regular polygon is found by multiplying the sum of all its sides by the radius of the inscribed circle, and dividing the product by 2; or we may at once multiply half the sum of all the sides by the radius of the inscribed circle, or half that radius by the sum of all the sides.

37. If the arcs subtended by the sides of a regular polygon, inscribed in a circle, are bisected, and chords drawn from the extremities of these arcs to the points of division, the new figure thus inscribed in the circle, is a regular polygon of twice the number of sides as the one first inscribed.

38. The circumference of a circle differs so little from the sum of all the sides of a regular inscribed polygon of a great number of sides,

(several thousands for instance) that, without perceptible error, the one may be taken for the other.

39. The circumferences of two circles are in proportion to the radii of these circles; that is, a straight line, as long as the circumference of the first circle, is as many times greater than a straight line as long as the circumference of the second circle, as the radius of the one is greater than the radius of the other.

40. The area of two circles are in proportion to the squares constructed upon their radii; that is, the area of the greater circle is as many times greater than the area of the smaller circle, as the area of the square upon the radius of the one, is greater than the area of the square upon the radius of the other.

41. The area of a circle is found by multiplying the circumference, given in rods, feet, inches, &c. by half the radius given in units of the same kind.

42. The circumference of a circle, whose radius is 1, is equal to the number 6,2831852, and the circumference of any other circle is found by multiplying the number 6,2831852 by the length of the radius.

43. The length of 1 degree in a circle, whose radius is 1, is equal to the number 0,0174533

The length of 1 minute	0,0002909
" " " 1 second	0,0000048
" " " 1 third	0,0000001

44. The length of an arc given in degrees, minutes, seconds, &c. is found by multiplying the degrees by 0,0174533, the minutes by 0,0002909, the seconds by 0,0000048, &c. then adding these products together, and multiplying their sum by the radius of the circle.

45. The area of a circle, whose radius is 1, is equal to 3,1415926 square units; and the area of any other circle is found by multiplying the number 3,1415926 by the square of the radius.

46. The area of a semicircle is found by dividing the area of the whole circle by 2.

47. The area of a quadrant is found by dividing the area of the whole circle by 4.

48. The area of a sector is found by multiplying the length of the arc by half the radius.

49. In order to find the area of a segment, we first draw two radii to the extremities of the arc of that segment; then calculate the area of the sector, formed by the two radii and that arc, and subtract from it the area of the triangle formed by the two radii and the chord of the segment; the remainder is the area of the segment.*

*

* The teacher ought now to ask his pupil to demonstrate these principles.

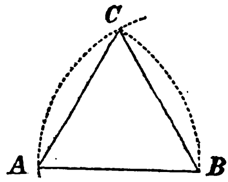
SECTION V.

APPLICATION OF THE FOREGOING PRINCIPLES TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

PART I.

Problems relative to the drawing and division of lines and angles.

PROBLEM I. *To construct an equilateral triangle upon a given straight line AB.*



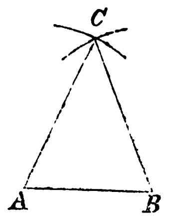
SOLUTION. Let AB be the given straight line.

1. From the point A as a centre, with the radius AB, describe an arc of a circle, and from the point B, with the same radius AB, another arc cutting the first.

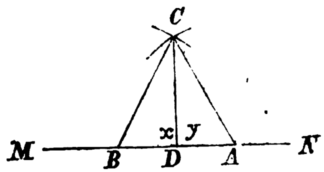
2. From the point of intersection C draw the lines AC, BC; the triangle ABC will be equilateral.

DEMONSTRATION. The three sides AB, AC, BC, of the triangles ABC, are all equal to each other; because they are radii of equal circles.

Remark. In a similar manner can an isosceles triangle be constructed upon a given basis.



PROBLEM II. *From a given point in a straight line, to draw a perpendicular to that line.*



I. SOLUTION. Let MN be the given straight line, and D the point in which another straight line is to be drawn perpendicular to it.

1. Take any distance BD on one side of the point D, and make DA equal to it.

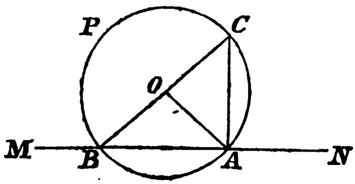
2. From the point B, with any radius greater than BD, describe an arc of a circle, and from the point A, with the same radius, another arc cutting the first.

3. Through the point of intersection C and the point D draw a straight line CD, which will be perpendicular to the line MN.

Demon. The three sides of the triangle BCD are equal to the three sides of the triangle ACD; each to each, viz.

the side BC equal to AC
" " BD " " DA
" " CD " " CD;

therefore the three angles in the triangle BCD are also equal to the three angles of the triangle ACD, each to each (page 44); and the angle x opposite to the side BC in the triangle BCD, is equal to the angle y opposite to the equal side AC in the triangle ACD; and as the two adjacent angles, which the line CD makes with the line MN, are equal to one another, the line CD is perpendicular to MN. (Definition of perpendicular lines, page 5).



II. Solution. Let MN be the given straight line, and A the point in which a perpendicular is to be drawn to it.

1. From a point O as a centre, with a radius OA, greater than the distance O from the straight line MN, describe the circumference of a circle.

2. Through the point B and the centre O, of the circle, draw the diameter BC.

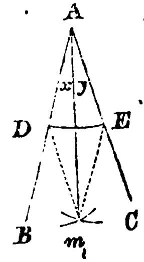
3. Through C and A draw a straight line, which will be perpendicular to the line MN.

Dem. The angle BAC, at the circumference, measures half as many degrees as the arc BPC intercepted by its legs (page

142). But the arc BPC is a semi-circumference; therefore the angle BAC measures a quadrant; consequently the angle BAC is a right angle, (page 138, Remark 3d), and the line AC is perpendicular to MN.

PROBLEM III. *To bisect a given angle.*

SOLUTION. Let BAC be the given angle.



1. From the vertex A, of the angle BAC, with a radius AE, taken at pleasure, describe an arc of a circle, and from the two points D and E, where this arc cuts the legs of the given angle, with the same radius describe two other arcs, cutting each other in the point m.

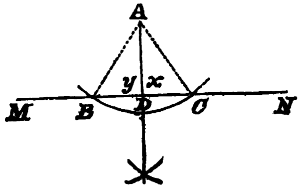
2. Through the point M, and the vertex of the given angle, draw a straight line Am, which will bisect the given angle BAC.

DEMON. The two triangles AmD, AmE, have the three sides in the one, equal to the three sides in the other, viz.

the side AD = to the side AE
" " mD = " " " mE
and " " Am = " " " Am;

consequently these two triangles are equal to each other; and the angle x opposite to the side mD, in the triangle AmD, is equal to the angle y, opposite to the equal side mE in the triangle AmE; therefore the angle BAC is bisected.

PROBLEM IV. *From a given point* without *a straight line to draw a perpendicular to that line.*



SOLUTION. Let A be the given point, from which a perpendicular is to be drawn to the line MN.

1. With any radius sufficiently great describe an arc of a circle.

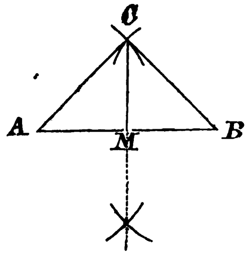
2. From the two points B and C, where this arc cuts the line MN, draw the straight lines BA, CA.

3. Bisect the angle BAC (see the last Problem), the line AD is perpendicular to the line MN.

DEMON. The two triangles ABD, ACD, have two sides AB, AD, in the one, equal to two sides AC, AD, in the other, each to each; (AC, AB, being radii of the same circle, and the side AD being common to both); and have the angles included by these sides also equal (because the angle BAC is bisected); therefore these two triangles are equal to one another (Query I, Sect. II); and the angle y, opposite to the side AB in the triangle ABD, is equal to the angle x, opposite to the equal side AC in the triangle ACD. Now as the two adjacent angles x and y, which the straight line AD makes with the straight line MN, are equal to each other, the line AD must be perpendicular to MN. (Def. of perpendicular lines.)

PROBLEM V. *To bisect a given straight line.*

SOLUTION. Let AB be the given straight line.



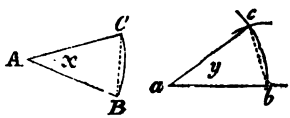
1. From A, with a radius greater than half of AB, describe an arc of a circle, and from B, with the same radius, another, cutting the first in the point C.

2. From the point C draw the perpendicular CM, and the line AB is bisected in M.

DEMON. The two right-angular triangles AMC, BMC, are equal, because the hypothenuse AC and the side CM in the one, are equal to the hypothenuse and the side CM in the other (page 55); and therefore the third side AM in the one, is also equal to the third side BM in the other; consequently the line AB is bisected in the point M.

PROBLEM VI. *To transfer a given angle.*

SOLUTION. Let x be the given angle, and A the point to which it is to be transferred.



1. From the vertex of the given angle, as a centre, with a radius taken at pleasure, describe an arc of a circle between the legs AB, AC.

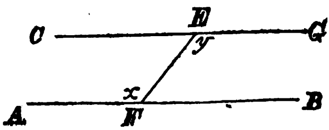
2. From the point a, as a centre, with the same radius, describe another arc $c\,b$.

3. Upon the last arc take a distance *b c* equal to the chord BC.

4. Through A and C draw a straight line; the angle *y* is equal to the angle *x*.

DEMON. The arcs BC, *b c*, are, by construction, equal to one another; therefore the angles *x* and *y*, at the centre, being measured by these arcs are also equal to one another (page 137, 1st).

PROBLEM VII. *Through a given point draw a line parallel to a given straight line.*

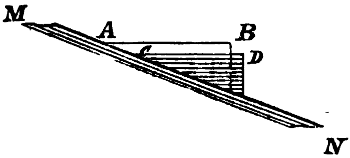


SOLUTION. Let E be the point, through which a line is to be drawn parallel to the straight line AB.

1. Take any point F, in the straight line AB, and join EF.

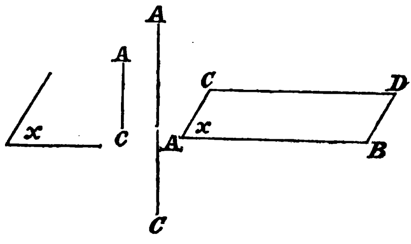
2. In E make an angle *y* equal to the angle *x*; the line EG extended is parallel to the line AB.

DEMON. The two straight lines CG, AB, are cut by a third line EF, so as to make the alternate angles *x* and *y* equal; therefore these two lines are parallel to each other (page 51, 1stly).



MECHANICAL SOLUTION. Take a ruler MN, and put it in such a position that a right-angled triangle, passing along its edge, as you see in the figure, will make with it in different points A, C, &c. the lines AB, CD &c. These lines are parallel to each other, because they are cut by the edge of the ruler at equal angles.*

PROBLEM VIII. *Two adjacent sides and the angle included by them being given, to construct a parallelogram.*



SOLUTION. Let AB and AC be the two sides of the parallelogram and x the angle included by them.

* This is a better way of drawing parallel lines than the common method by a parallel ruler, which is seldom very accurate, on account of the instrument being frequently out of order, and the great steadiness of hand required in the use of it.

1. Make an angle equal to x.

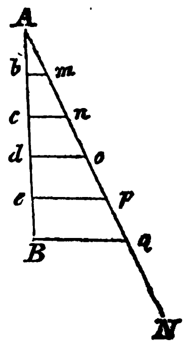
2. Make the leg AB of that angle equal to AB, and the legs AC equal to AC.

3. Through the point C draw CD parallel to AB, and through B, the line BD parallel to AC; the quadrilateral ABCD is the required parallelogram.

DEMON. The opposite sides of the quadrilateral ABCD are parallel to each other; therefore the figure is a parallelogram. (See Def. page 7).

PROBLEM IX. *To divide a given line into any number of equal parts.*

I. SOLUTION. Let AB be the given line, and let it be required to divide it into five equal parts.



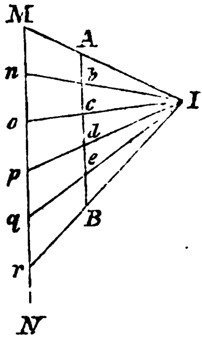
1. From the point A draw an indefinite straight line, making any angle you please with the line AB.

2. Take any distance A m, and measure it off 5 times upon the line AN.

3. Join the last point of division q, and the extremity B of the line AB.

4. Through m, n, o, p, q, draw the straight lines $b\,m$, $c\,n$, $d\,o$, $e\,p$, parallel to B q; the line AB is divided into 5 equal parts.

The demonstration follows immediately from Query 14, Sect. II.



II. Solution. Let AB be the given straight line, which is to be divided into 5 equal parts.

1. Draw a straight line MN, greater than AB, parallel to AB.

2. Take any distance M *n*, and measure it off 5 times upon the line MN.

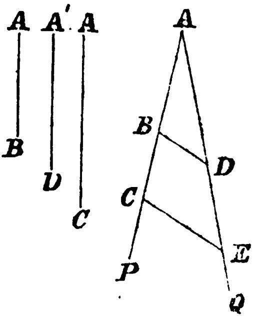
3. Join the extremities of both the lines M *r* and AB, by the straight lines MA, *r* B, which will cut each other when sufficiently far extended in a point I.

4. Join I *n*, I *o*, I *p*, I *q*, the line AB is divided into 5 equal parts, viz. A *b*, *a d*, *b c*, *c d*, *d e*, *e* B.

Demon. The triangles A *b* I, *b c* I, *c d* I, *d e* I, *e* BI are similar to the triangles M *n* I, *n o* I, *o p* I, *p q* I, *q r* I, each to each; because the line AB is drawn *parallel* to M *r* (Query 16, Sect. II.); and as the bases M *n*, *n o*, *o p*, *p q*, *q r*, of the latter triangles are all equal to one another, the bases A *b*, *b c*, *c d*, *d e*, *e* B, of the former triangles must also be equal to one another.

Remark. If it were required to divide a line into two parts which shall be in a given ratio, for instance, as 2 to 3, you need only, as before, take 5 equal distances upon the line MN, and then join the point I to the second and last point of division; the line AB will, in the point *c*, be divided in the ratio of 2 to 3. In a similar manner can any given straight line be divided into 3, 4, 5, &c. parts, which shall be to each other in a given ratio.

PROBLEM X. ***Three lines being given to find a fourth one, which shall be in a geometrical proportion with them.***



SOLUTION. Let AB, AC, AD, be the given straight lines, which are three terms of a geometrical proportion, to which the fourth term is wanting. (See Theory of Proportions, Principle 6th, page 75.)

1. Draw two indefinite straight lines AP, AQ, making with one another any angle you please.

2. Upon one of these lines measure off the two distances AB, AC, and on the other the distance AD.

3. Join BD, and through C draw CE parallel to BD; the line AE is the fourth term in the geometrical proportion

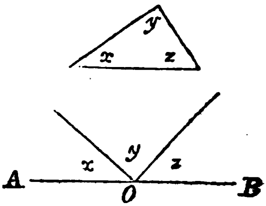
$$AB : AC = AD : AE.$$

DEMON. The triangle ABD is similar to the triangle ACE, from which it may be considered as cut off by the line BD being drawn parallel to CE (Query 16, Sect. II.); and as in similar triangles the corresponding sides are in a geometrical proportion (page 82, 4thly,) we have

$$AB : AC = AD : AE$$

PROBLEM XI. *Two angles of a triangle being given, to find the third one.*

SOLUTION. Let x and y be the two given angles of the triangle, and let it be required to find the third angle z.

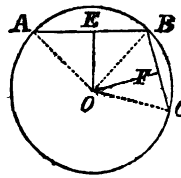


In any point O of an indefinite straight line AB, make two angles x y, equal to the two given angles of the triangle; the remaining angle z is equal to the angle z in the triangle.

DEMON. The sum of the three angles x, z, y, in the triangle is equal to two right angles (Query 14, Sect. I.), and the sum of the three angles x, y, z made in the same point O, and the same side of the straight line AB is also equal to two right-angles (page 23); and as the angles x and y are made equal to the angle x and y in the triangle, the remaining angle z is also equal to the remaining angle z in the triangle.

Remark, If instead of the angles themselves, their measure were given in degrees, minutes, seconds, &c. you need only subtract the sum of the two angles from 180 degrees, which is the measure of two right angles; the remainder is the angle sought.

PROBLEM XII. *Through three given points, which are not in the same straight line, to describe the circumference of a circle.*



Solution. Let A, B, C, be the three points, through which it is required to pass the circumference of a circle.

1. Join the three points A, B, C by the straight lines AB, BC.

2. Bisect the lines AB, BC.

3. In the points of bisection E and F, erect the perpendiculars EO, FO, which will cut each other in a point O.

4. From the point O as a centre, with a radius equal to the distance AO, describe the circumference of a circle, and it will pass through the three points A, B, C.

Demon. The two points A and B are at an equal distance from the foot of the perpendicular EO; therefore AO and BO are equal to one another (page 53, 5thly); and for the same reason is BO equal to OC; because the points B and C are at an equal distance from the foot of the perpendicular FO; and as the three lines AO, BO, CO are equal to one another, the three points A, B, C, must necessarily lie in the circumference of the circle described with the radius AO.

Problem XIII. *To find the centre of a circle or of a given arc.*

Solution. Let the circle in the last figure be the given one.

1. Take any three points A, B, C in the circumference, and join them by the chords AB, BC.

2. Bisect each of these chords, and in the points of bisection erect the perpendiculars EO, FO; the point O, in which these perpendiculars meet each other, is the centre of the circle.

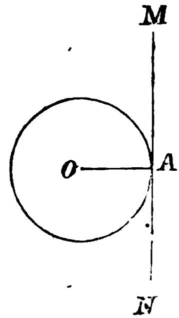
In precisely the same manner can the centre of an arc be found.

The demonstration is exactly the same as in the last problem.

Problem XIV. *In a given point in the circumference of a circle, to draw a tangent to that circle.*

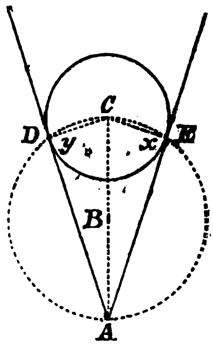
Solution. Let A be the given point in the circumference of the circle.

Draw the radius AO, and at the extremity A, perpendicular to it, the line MN; and it is a tangent to the given circle.



Demon. The line MN, being drawn at the extremity A of the radius, and perpendicular to it, touches the circumference in only one point (page 138).

Problem XV. *From a point* without *a circle, to draw a tangent to the circle.*



Solution. Let A be the given point from which a tangent is to be drawn to the circle.

1. Join the point A and the centre C of the given circle.

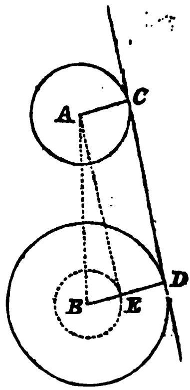
2. From the middle of the line AC as a centre, with a radius equal to BC = AB, describe the circumference of a circle.

3. Through the points E and D, where this circumference cuts the circumference of the given circle, draw the lines AD, AE; and they are tangents to the given circle.

Demon. Join DC EC. The angles x and y, being both angles at the circumference of the circle whose centre is B, measure each half as many degrees as the arc on which their legs stand. Both the angles x and y have their legs stand on the diameter AC of the circle B; therefore each of these angles measures half as many degrees as the semi-circumference, that is, 90 degrees (page 138, rem. 3d); consequently they are both right angles, and the lines AE and DA, being perpendicular to the radii CE, DC, are both tangents to the circle C.

Remark. From a point without a circle you can always draw *two* tangents to the same circle.

Problem XVI. *To draw a tangent common to two given circles.*



SOLUTION. Let A and B be the centres of the given circles, and let it be required to draw a tangent which shall touch the two circles on the same side.

1. Join the centres of the two given circles by the straight line AB.

2. From B, as a centre, with a radius equal to the difference between the radii of the given circles, describe a third circle.

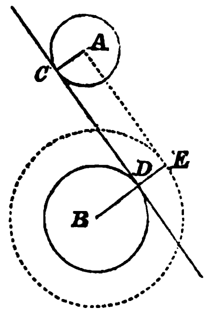
3. From A draw a tangent AE to that circle (see the last problem).

4. Draw the radius BE, and extend it to D.

5. Draw the radius AC parallel to BD.

6. Through C and D draw a straight line, and it will be a tangent common to the two given circles.

DEMON. The radius AC being equal and parallel to ED, it follows that ACED is a parallelogram (page 60); and because the tangent AE is *perpendicular* to the radius BE (page 139, 1st), CD is perpendicular to BD; consequently also to AC (because AC is parallel to BD); and the line CD, being perpendicular to both the radii AC, BD, is a tangent common to the two given circles.



If it be required to draw a tangent common to two given circles, which shall touch them on opposite sides, then

1. From B, as a centre, with a radius equal to the *sum* of the radii of the given circles, describe a third circle.

2. From A draw a tangent AE to that circle.

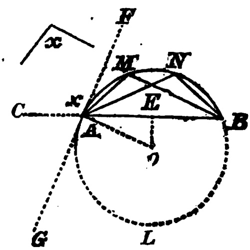
3. Join BE, cutting the given circle in D.

4. Draw AC parallel to BE.

5. Through C and D, draw a straight line, and it is the required tangent, touching the circles on opposite sides.

The demonstration is the same as the last.

PROBLEM XVI. *Upon a given straight line to describe a segment of a circle, which shall contain a given angle; that is, a segment, such that the inscribed angles, having their vertices in the arc of the segment and their legs standing on its extremities, shall each be equal to a given angle.*



SOLUTION. Let AB be the given line, and x the given angle.

1. Extend AB towards C.

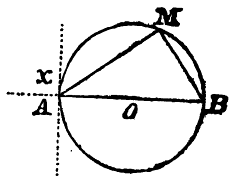
2. Transfer the angle x to the point A.

3. Bisect AB in E.

4. From the points A and E draw the lines AO and EO, respectively perpendicular to FG and AB.

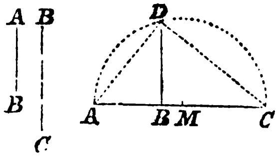
5. From the point O, the intersection of these perpendiculars, as a centre, with a radius equal to OA describe a circle; AMNB is the required segment.

DEMON. The line AF being by construction perpendicular to the radius OB, is a tangent to the circle (page 138); and the angle GAB, formed by that tangent and the chord AB, is equal to either of the angles AMB, ANB, &c. that can be inscribed in the segment AMNB; because the angle GAB measures half as many degrees as the arc ALB (page 140), and each of the angles AMB, ANB &c. at the circumference, having its legs standing on the extremities of the chord AB, measures also half as many degrees as the arc ALB (page 141); and as the angle GAB is equal to the angle x, GAB and x being opposite angles at the vertex (Query 5, Sect. I.), each of the angles AMB, ANB &c. is also equal to the given angle x.



Remark. If the angle x is a right angle, the segment AMB is a semicircle, and the chord AB a diameter. To finish the construction, you need only from the middle of the line AB as a centre, with a radius equal to OA, describe a semicircle, and it is the required segment; for the angle AMB at the circumference measuring half as many degrees as the semi-circumference AB, on which its legs stand, is a right angle.

PROBLEM XVIII. *To find a mean proportional (see note 2d to page* 89*) to two given straight lines.*



SOLUTION. Let AB, BC be the two given lines.

1. Upon an indefinite straight line, take the two distances AB, BC.

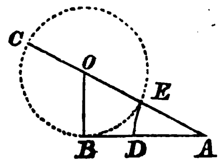
2. Bisect the whole distance AC, and from M, the middle of AC, with a radius equal to AM, describe a semi-circumference.

3. In B erect a perpendicular to the diameter AC, and extend it until it meets the semi-circumference in D; the line DB is a mean proportional between the lines AB and BC.

DEMON. The triangle ADC is right-angular in D; because the angle ADC is inscribed in a semicircle (see the remark to the last problem); and the perpendicular DB dropped from the vertex D, of the right angle to the hypothenuse, is a mean proportional between the two parts AB, BC, into which it divides the hypothenuse; (page 89, 1st) therefore we have the proportion

$$AB : BD = BD : BC.$$

PROBLEM XIX. *To divide a given straight line into two such parts, that the greater of them shall be a mean proportional between the smaller part and the whole of the given line.*



SOLUTION. Let AB be the given straight line.

1. At the extremity B of the given line, erect a perpendicular, and make it equal to half of the line AB.

2. From O, as a centre, with a radius equal to OB, describe a circle.

3. Join the centre O of that circle, and the extremity A of the given line, by the straight line AO.

4. From AB cut off a distance AD equal to AE; then AD is a mean proportional between the remaining part BD, and the whole line AB; that is, you have the proportion

$$AB : AD = AD : BD.$$

DEMON. Extend the line AO until it meets the circumference in C. Then the radius OB, being perpendicular to the line AB, we have from the same point A, a tangent AB, and another line AC drawn cutting the circle; therefore we have the proportion

$$AC : AB = AB : AE;$$

for the tangent AB is a mean proportional between the whole line AC, and the part AE without the circle (see page 143, Query 13.)

Now in every geometrical proportion you can add or subtract the second term once or any number of times from the first term, and the fourth term the same number of times from the third term, without destroying the proportion (see the note at the end of the book). According to this principle you have

$$AC - AB : AB = AB - AE : AE;$$

that is, the line AC less the line AB, is to the line AB, as the line AB less EA, is to the line AE. But AC less AB is the same, as the line AC less the diameter CE; (because the *radius* of the

circle is by construction equal to *half* the line AB); and AB less AE, is the same as AB less AD; (because AD is made equal to AE); therefore you may write the above proportion also

AE : AB = BD : AE,* or also

AD : AB = BD : AD;

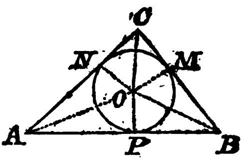
and because in every geometrical proportion the order of the terms may be changed in both ratios, (principle 1, of Geom. Prop. page 67), you can change the last proportion into

AB : AD = AD : BD:

that is, the part AD of the line AB is a mean proportional between the whole line AB and the remaining part BD.

PROBLEM XX. *To inscribe a circle in a given triangle.*

SOLUTION. Let the given triangle be ABC.



1. Bisect two of the angles of the given triangle; for instance the angles at C and B, by the lines CO, BO.

2. From the point O, where these lines cut each other, drop a perpendicular to any of the sides of the given triangle.

3. From O, as a centre, with the radius OP, equal to the length of that perpendicular; describe a circle, and it will be inscribed in the triangle ABC.

DEMON. From O drop the perpendiculars OM, ON upon the two sides BC, AC of the given triangle. The angle OCM is, by construction, equal to the angle OCN; (because the

---

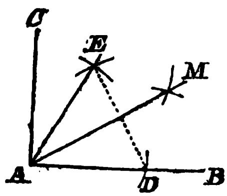
* AC less the diameter CE, being equal to AE; and BA less AD, equal to BD.

angle ACB is bisected by the line CO); and CMO, CNO being right angles, the angles COM and CON are also equal to one another; (because when two angles in one triangle are equal to two angles in another, the third angles in these triangles are also equal) therefore the two triangles CMO, CNO have a side CO, and the two adjacent angles in the one, equal to the same side CO, and the two adjacent angles in the other; consequently these two triangles are equal to one another; and the side OM, opposite to the angle OCM in the one, is equal to the side ON, opposite to the equal angle OCN in the other. In the same manner it may be proved that because the triangle OMB is equal to the triangle OPB, the perpendicular OM is also equal to OP; and as the three perpendiculars OM, ON, OP are equal to one another, the circumference of a circle described from the point O as a centre, with a radius equal to OP, passes through the three points M, N, P; and the sides AB, BC, AC of the given triangle, being perpendicular to the radii OP, OM, ON, are tangents to the inscribed circle (page 138).

PROBLEM XXI. *To circumscribe a circle about a triangle.*

This problem is the same as to make the circumference of a circle pass through three given points. (See Problem XII.)

PROBLEM XXII. *To trisect a right angle.*



SOLUTION. Let BAC be the right angle which is to be divided into three equal parts.

1. Upon AB take any distance AD, and construct upon it the equilateral triangle ADE. (Problem I.)

2. Bisect the angle DAE by the line AM (Problem III.); and the right angle BAC is divided into the three equal angles CAE, EAM, MAB.

Demon. The angle BAE being one of the angles of an equilateral triangle, is one third of two right angles (pages 44 and 36), and therefore *two* thirds of *one* right angle; consequently CAE is *one* third of the right angle BAC; and since the angle BAE is bisected by the line AM, the angles EAM, MAB, are each of them also equal to one third of a right angle; and are therefore equal to the angle CAE and to each other.

* * *

PART II.

Problems relative to the transformations of geometrical figures.

Problem XXIII. *To transform a given quadrilateral figure into a triangle of equal area, whose vertex shall be in a given angle of the figure, and whose base in one of the sides of the figure.*

Fig. I.

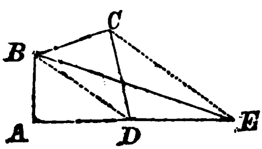
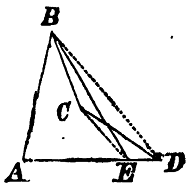


Fig. II.



Solution. Let ABCD (fig. I. and II.) be the given quadrilateral; the figure I. has all its an-

gles outwards, and the figure II. has one angle BCD inwards; let the vertex of the triangle, which shall be equal to it, fall in B.

1. Draw the diagonal BD (fig. I. and II.) and from C, parallel to it, the line CE.

2. From E, where the line CE cuts AD (fig. II.), or its further extension (fig. I.), draw the line EB; the triangle ABE is equal to the quadrilateral ABCD.

Demon. The area of the triangle BCD (fig. I. and II.) is equal to the area of the triangle BDE; because these two triangles are upon the same basis BD, and between the same parallels BD, CE (page 110, 3dly); consequently (fig. I.) the *sum* of the areas of the two triangles ABD and BDC is equal to the sum of the areas of the two triangles ABD, BDE; that is, the area of the quadrilateral ABCD is equal to the sum of the areas of the two triangles ABD, BDE, which is the area of the triangle ABC.

And in figure II. the *difference* between the areas of the two triangles ABD, BCD; that is, the quadrilateral ABCD is equal to the difference between the triangles ABD, EBD, which is the triangle ABE.

Problem XXIV. *To transform a given pentagon into a triangle, whose vertex shall be in a given angle of the pentagon, and whose base upon one of its sides.*

Solution. Let ABCDE (fig. I. and II.) be the given pentagon; let the vertex of the triangle, which is to be equal to it, be in C.

Fig. I.

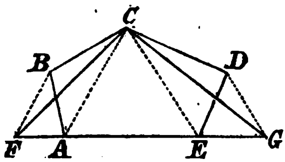
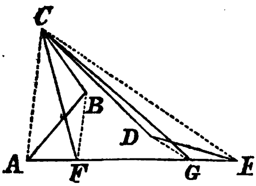


Fig. II.



1. From C draw the diagonals CA, CE.

2. From B draw BF parallel to CA, and from D draw DG parallel to CE.

3. From F and G, where these parallels cut AE or its further extension, draw the lines CF, CG, CFG is the triangle sought.

Demon. In both figures we have the area of the triangle CBA equal to the area of the triangle CFA; because these two triangles are upon the same basis CA, and between the same parallels AC, FB; and for the same reason is the area of the triangle CDE equal to the area of the triangle CGE; therefore in figure I. the sum of the areas of the three triangles CAE, CBA, CDE, is equal to the sum of the areas of the triangles CAE, CFA, CGE; that is, the area of the pentagon ABCDE is equal to the area of the triangle CFG: and in figure II. the difference between the area of the triangle CAE and the sum of the areas of the two triangles CBA, CDE, is equal to the difference between the area of the same triangle CAE and the sum of the areas of the two triangles CFA, CGE; that is, the area of the pentagon ABCDE is equal to the area of the triangle CFG.

Problem XXV. *To convert any given figure into a triangle, whose vertex shall be in a given*

angle of the figure, and whose base is one of the sides.

FIG. I.

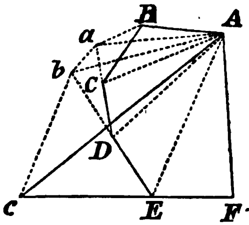
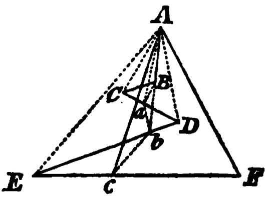


FIG. II.



Let ABCDEF (fig. I. and II.) be the given figure (in this case a hexagon,) and A the angle in which the vertex of the required triangle shall be situated. For the sake of perspicuity I shall enumerate the angles and sides of the figure from A, and call the first angle A, the second B, the third C, and so on; further, AB the first side, BC the second, DE the third, and so on. We shall then have the following general solution.

1. From A to all the angles of the figure draw the diagonals AC, AD, AE, which, according to the order in which they stand here, call the first, second, and third diagonal.

2. Draw from the second angle B, a line B *a* parallel to the first diagonal AC; from the point *a*, where the parallel meets the third side CD, (fig. II.), or its further extension (fig. I.), draw a line *a b* parallel to the second diagonal AD; and from the point *b*, where this meets

the fourth side DE (fig. II.) or its further extension (fig. I.), draw another line *b c* parallel to the third diagonal.

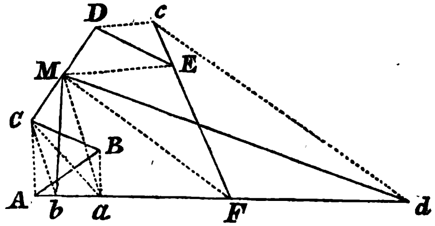
3. When in this way you have drawn a parallel to every diagonal, then from the last point of section of the parallels and sides, (in this case *c*), draw the line *c* A; A *c* F is the required triangle, whose vertex is in A, and whose base is in the side EF.

The demonstration is similar to the one given in the two last problems (pages 203, 204.) First, each of the hexagons is converted into the pentagon A *a* DEF, then the pentagon A *a* DEF into a quadrilateral A *b* EF, and finally the quadrilateral into the triangle A *c* F. The areas of these figures are evidently equal to one another; for the areas of the triangles, which, by the above construction, are successively cut off from these figures, are equal to the areas of the new triangles, which are successively added to them. (See the demonstration of the last problem.)

Remark. Although the solution given here, is only intended for a hexagon, yet it may easily be applied to every other rectilinear figure. All depends upon the substitution of one triangle for another by means of parallel lines; in which you have only to take care, that one side of the triangle substituted, be in a side of the figure, or in its further extension; because by these means the number of sides will be diminished. Moreover it is not absolutely necessary actually to draw the parallels; it is only requisite to note the points in which they cut the sides or their further extension; because all depends upon the determination of these points of section.

Problem XXVI. *To transform any given figure into a triangle whose vertex shall be in a certain point, in one of the sides of the figure, or*

within it, and whose base shall be in a given side of the figure.

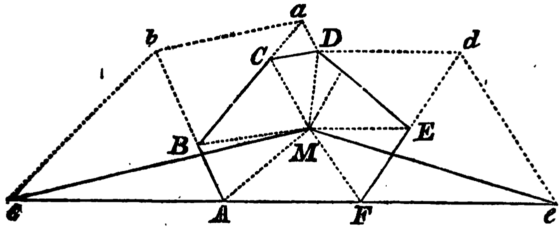


SOLUTION. 1st *Case.* Let ABCDEF be a hexagon, which is to be transformed into a triangle; let the vertex of the triangle be in the point M in the side CD, and the base in AF.

1. In the first place, omit the angle ABC by drawing B *a* parallel to CA, and joining C *a*; the triangle CB *a*, substituted for its equal the triangle AB *a*, (for these two triangles are upon the same basis *a* B, and between the same parallels CA, B *a*), transforms the hexagon ABCDEF into the pentagon *a* CDEF.

2. Draw the lines M *a*, MF, and the pentagon *a* CDEF is divided into the three figures, *viz.* the triangle M *a* F, the quadrilateral MDEF on the right, and the triangle MC *a* on the left.

3. Transform the quadrilateral MDEF and the triangle MC *a* into the triangles M *d* F, M *b a*, so that the basis may be in AF (see the last problem); the triangle M *b d* is equal to the given hexagon.



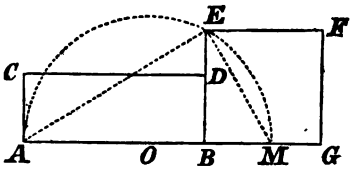
3d Case. Let ABCDEF be the given figure; let the vertex of the required triangle be situated in the point M within the figure, and let the base be in AF.

1. From M to any angle of the figure, say D, draw the line MD, and draw the lines MA, MF, by which means the figure ABCDEF is divided into the triangle MAF, and the figures MDCBA, MDEF.

2. Then transform MDCBA and MDEF into the triangles M*c*A, M*e*F, whose bases are in AF; the triangle *c*M*e* is equal to the figure ABCDEF.

The demonstration follows from those of the last three problems.

PROBLEM XXVII. *To transform a given rectangle into a square of equal area.*



SOLUTION. Let ABCD be the given rectangle.

1. Extend the greater side AB of the rectangle, making BM equal to BD.

2. Bisect AM in O, and from the point O, as a centre, with a radius AO, equal to OM, describe a semicircle.

3. Extend the side BD of the rectangle, until it meets the circle in E.

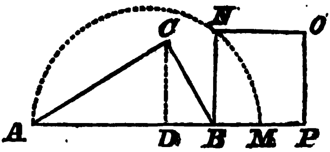
4. Upon BE construct the square BEFG, which is the square sought.

DEMON. The perpendicular BE is a mean proportional between AB and BM (see problem XVIII.); therefore we have the proportion

$$AB : BE = BE : BM;$$

and as in every geometrical proportion the product of the means equals that of the extremes (theory of prop. principle 8, page 76), we have the product of the side BE multiplied by itself, equal to the product of the side AB of the parallelogram, multiplied by the adjacent side BD (equal to BM.) But the first of these products is the area of the square BEFG, and the other is the area of the rectangle ABCD; therefore these two figures are, in area, equal to one another.

PROBLEM XXVIII. *To transform a given triangle into a square of equal area.*



SOLUTION. Let ABC be the given triangle, AB its base, and CD its height.

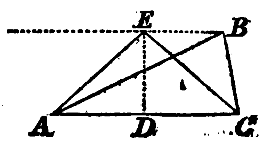
1. Extend AB by half the height CD.
2. Upon AM as a diameter describe a semicircle.
3. From B draw the perpendicular BN, which is the side of the square sought.

DEMON. From the demonstration in the last problem, it follows that the square upon BN is equal to the rectangle, whose base is AB, and whose height is BM (half the height of the triangle ABC). But the triangle ABC is equal to a rectangle upon the same base AB, and of half the height CD, (page 109, 1st); therefore the area of the square BNOP is also equal to the area of the triangle ABC.

Remark. It appears from this problem, that every rectilinear figure can be converted into a square of equal area. It is only necessary to convert the figure into a triangle (according to the rules given in the problems 23, 24, 25), and then that triangle into a square.

PROBLEM XXIX. *To convert any given triangle into an isosceles triangle of equal area.*

SOLUTION. Let ABC be the given triangle, which is to be converted into an isosceles one.



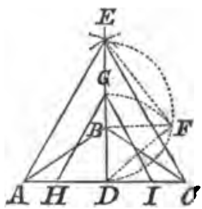
1. Bisect the base AC in D, and from D draw the perpendicular DE.

2. From the vertex B of the given triangle, draw BE parallel to the base AC.

3. From the point E, where this parallel meets the perpendicular, draw the straight lines EA, EC; EAC is the isosceles triangle sought.

DEMON. The triangles AEC and ABC are upon the same basis AC, and between the same parallels (page 110, 3dly).

PROBLEM XXX. *To convert a given isosceles triangle into an equilateral one of equal area.*



SOLUTION. Let ABC be the given isosceles triangle.

1. Upon the base AC of the given triangle draw the equilateral triangle AEC (problem I.); and through the vertices E, B, of the two triangles draw the straight line EB, which evidently is perpendicular to AC, and bisects the last line in D.*

2. Upon ED describe the semicircle EFD, and from B draw the perpendicular BF, which meets the semicircle in F.

3. From D, with the radius DF, describe an arc FG, cutting the line DE in G.

4. From G draw the lines GH, GI, parallel to the sides of the equilateral triangle; HGI is the equilateral triangle sought.

DEMON. Since the line GH is parallel to AE, and GI parallel to EC, the angle GHI is equal to the angle EAI, and the angle GIH to the angle ECH (page 33). Thus the two triangles GHI, ACE have two angles GHI, GIH in the one, equal to two angles EAC, ECA in the other, each to each; consequently they are similar to each other (page 87, 1st); and the triangle GHI must also be equilateral. Suppose the lines DF and EF drawn; then DF, and consequently also GD (its equal) is a mean proportional between DE and DB; for the triangle EDF is right-angular (see the remark page 198) in F, and if from the vertex of the right angle the perpendicular FB is dropped upon the hypothenuse, the side DF of the triangle is a mean proportional between the hypothenuse ED and the part BD of it, between the foot of the perpendicular and the extremity D (see page 89, 2dly); consequently we shall have the proportion

$$ED : DF = DF : BD,$$

and as DG is by construction made equal to DF

$$ED : DG = DG : BD \ldots\ldots\ldots\ldots (\mathrm{I.})$$

Moreover in the two similar triangles ADE, HDG the corresponding sides are proportional (page 82, 4thly); therefore we have the proportion

$$ED : DG = AD : HD \ldots\ldots\ldots\ldots (\mathrm{II.})$$

This last proportion has the first ratio common with the first proportion; consequently the two remaining ratios are in a geometrical proportion (Theory of prop. prin. 3d); that is, we have

$$AD : HD = DG : BD;$$

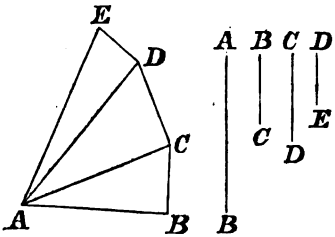
and as in every geometrical proportion the product of the means is equal to that of the extremes (Theory of prop. princ. 8th), we have HD multiplied by DG equal to AD multiplied by BD; consequently also *half* the product of the line HD, multiplied by the line DG, equal to half the product of the line AD multiplied by BD. But half the product of the line HD, multiplied by DG, is the area of the triangle HDG; because the triangle HDG is right-angular in D, therefore if HD is taken for the basis, DG

is its height; and for the same reason half the product of the line AD multiplied by BD, is the area of the triangle ADB; consequently the areas of the two triangles, ADB, and HDG, are equal to one another; and because the triangle ADG is equal to the triangle IDG, and the triangle ABD to the triangle CBD, the area of the whole triangle HIG is equal to the area of the whole triangle ABC; therefore the triangle HGI is the required equilateral triangle, which is equal, in area, to the given isosceles triangle ABC.

Remark. If BD be greater than ED, then the perpendicular BF does not meet the semicircle. In this case it will merely be necessary to describe the semicircle on BD, and from E to draw the perpendicular. In this case the points H, I, will not be situated in the line AC; but in its further extension.

Remark II. From this and the preceding problems it appears how any figure may be converted into an equilateral triangle; for it is only necessary first to convert the figure into a triangle, this triangle into an isosceles triangle, and the isosceles triangle into an equilateral one.

Problem XXXI. *To describe a square, which in area shall be equal to the sum of several given squares.*



Solution. Let AB, BC, CD, DE, be the sides of four squares; it is required to find a square which shall be equal to the sum of these four squares.

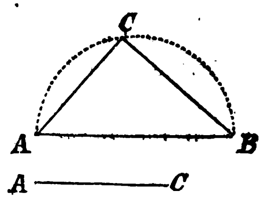
1. At the extremity B of the line AB draw a perpendicular equal to BC, and join AC.

2. At the extremity C of the line AC draw a perpendicular equal to CD, and join AD.

3. At the extremity D of the line AD draw a perpendicular equal to DE, and join AE; the square upon AE is in area equal to the sum of the four squares upon the lines AB, BC, CD, DE.

DEMON. The square upon the hypothenuse AC of the right-angular triangle ABC, is equal to the sum of the squares upon the two sides AB, BC, (Query 6, Sect. III.); and for the same reason is the square upon AD equal to the sum of the squares upon CD and AC; consequently also to the squares upon CD, CB and AB (the square upon AC being equal to the squares upon CB and AB); and finally the square upon AE is equal to the sum of the squares upon ED and AD; or, which is the same, to the sum of the squares upon DE, CD, CB and AB.

PROBLEM XXXII. *To describe a square which shall be equal to the difference of two given squares.*



SOLUTION. Let AB, AC, be the sides of two squares.

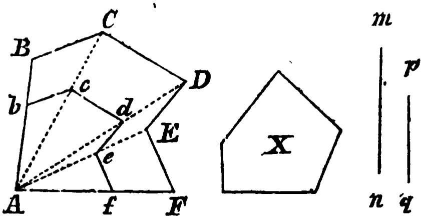
1. Upon the greater side AB, as a diameter, describe a semicircle.

2. From A, within the semicircle, draw the line AC equal to the given one, and join BC; then CB is the side of the square sought.

DEMON. The triangle ABC is right-angular in C (see the remark page 198), and in every right-angular triangle, the square

upon one of the sides, which include the right angle, is equal to the difference between the squares upon the hypothenuse and the other side.

PROBLEM XXXIII. *To transform a given figure in such a way, that it may be similar to another figure.*



SOLUTION. Let X be the given figure and ABCDEF the one to which it is to be similar.

1. Convert the figure ABCDEF into a square (remark, page 211), and let its side be $m\,n$, so that the area of the square upon $m\,n$ is equal to the area of the figure ABCDEF; convert also the figure X into a square, and let its side be $p\,q$, so that the area of the square upon $p\,q$ shall be equal to the area of the figure X.

2. Take any side of the figure ABCDEF, say AF; to the three lines $m\,n$, $p\,q$, AF, find a fourth proportional (problem X), which cut off from AF; let Af be this fourth proportional, so that we have the proportion

$$m\,n : p\,q = \text{AF} : \text{A}f.$$

3. Then draw the diagonals AE, AD, AC, and the lines $f\,e$, $e\,d$, $d\,c$, $c\,b$, parallel to the lines

FE, ED, DC, CB; then A $b\,c\,d\,e\,f$ will be the required figure, which in area is equal to the figure X, and is similar to the figure ABCDEF.

DEMON. It may be easily proved that the figure A $b\,c\,d\,e\,f$ is similar to ABCDEF. Further, we know that the areas of the two similar figures, ABCDEF, A $b\,c\,d\,e\,f$, are to each other, as the areas of the squares upon their corresponding sides AF, A f, (see page 126); which may be expressed

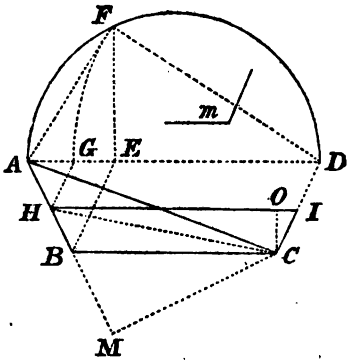
$$\text{ABCDEF} : \text{A}\,bcdef = \text{AF} \times \text{AF} : \text{A}f \times \text{A}f;$$

and as the sides AF and A f are (by construction 2) in proportion to the lines $m\,n$, $p\,q$, the figures ABCDEF, A $b\,c\,d\,e\,f$, will also be in proportion to the squares of these sides; that is, we shall also have the proportion

$$\text{ABCDEF} : \text{A}\,bcdef = mn \times mn : pq \times pq.$$

This proportion expresses, that the area of the figure ABCDEF is as many times greater than the area of the figure A $b\,c\,d\,e\,f$, as the area of the square upon the line $m\,n$ is greater than the area of the square upon the line $m\,q$; therefore as the area of the figure ABCDEF is, by construction 1, equal to that of the square upon the line $m\,n$, the area of the figure A $b\,c\,d\,e\,f$ is equal to that of the square upon the line $p\,q$. But the area of the square upon $p\,q$ is, by construction 1, equal to that of the figure X.; therefore the area of the figure A $b\,c\,d\,e\,f$ is also equal to that of the figure X.; consequently the figure A $b\,c\,d\,e\,f$ is the one required.

PROBLEM XXXIV. *Upon the base of a given triangle to describe a trapezoid, one of the parallel sides of which shall be the basis of the given triangle, and one of whose angles at the basis shall also be one of the angles of the triangle and the other equal to a given angle.*



Solution. Let ABC be the given triangle, which is required to be converted into a trapezoid, one of whose parallel sides is the base of the triangle; further, let the angle ABC of the triangle be also one of the angles of the trapezoid, and the other angle at C, equal to the given angle *m*.

1. On BC make the angle BCD equal to the angle *m* (problem VI.)

2. From the vertex A of the triangle, draw the line AD parallel to BC, which meets CD in D, and from B draw the line BE parallel to CD, which meets AD in E.

3. Upon AD describe a semicircle AFD, and from E draw the perpendicular EF, meeting the semicircle in F.

4. From D, with the radius DF, describe the arc FG, meeting the line AD in G.

5. From G draw the line GH, parallel to DC, and from the point H, in which it meets AB, the line HI, parallel to BC; then BHIC is the trapezoid sought.

DEMON. Suppose AF drawn. The triangle AFD is right-angular in F (remark page 198); therefore the side DF is a mean proportional between the whole hypothenuse AD and the part ED (page 89, 2dly); that is, we have the proportion

$$AD : DF = DF : DE,$$

or, since DF is equal to DG,

$$AD : DG = DG : DE;$$

and as in every geometrical proportion the second term can be subtracted from the first term, if the fourth be also subtracted from the third term, without destroying the proportion, we have also

$$AD - DG : DG = DG - DE : DE,$$

and by changing the order of the mean terms (principle 2d of geom. prop. page 67)

$$AD - DG : DG - DE = DG : DE,$$

which may also be written

$$AG : GE = HI : BC;$$

(because AD less DG is AG; DG less DE is GE; DG is by construction equal to HI, and DE is equal to BC.)

Further, from the two similar triangles AHG, ABE, (the line HG is drawn parallel to BE), we have the proportion

$$AB : AH = AE : AG;$$

and consequently,

$$AB - AH : AH = AE - AG : AG;$$

$$\text{or, } HB : AH = GE : AG;$$

(because AB — AH is HB, and AE — AG is GE).

And by changing the order of the terms in each ratio (principle 1st of geom. prop. page 67)

$$AH : HB = AG : GE.$$

This last proportion and the above one, AG : GE = HI : BC, have the ratio AG : GE common; consequently we have the new proportion

$$AH : HB = HI : BC.$$

Suppose HC drawn. The two triangles AHC, HBC, have the same height MC, consequently their areas are to each other as the bases AH, HB, (see page 111, 6thly), and we have the proportion

triangle AHC : triangle BCH = AH : HB.

For the same reason we have

triangle HIC : triangle BCH = HI : BC;

(because the two triangles HIC, BCH, being, by construction 5, between the same parallels, have the same height CO); and as we have the proportion

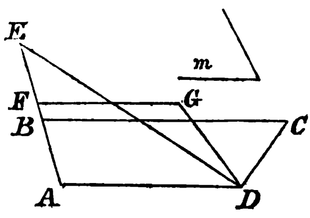
AH : HB = HI : BC,

we have also

triangle AHC : triangle BCH = triangle HIC : triangle BCH.

This last proportion expresses, that the area of the triangle AHC is as many times greater than the area of the triangle BCH, as the area of the triangle HIC is greater than the area of the same triangle BCH; consequently the triangle AHC is in area equal to the triangle HIC; and therefore, adding to each of the two equal triangles AHC, HIC, the same triangle HBC, the area of the whole triangle ABC is equal to that of the trapezoid HIBC.

Problem XXXIV. *To convert a given trapezoid into another, having one side, and one of the angles adjacent to this side, common with the former, and the other angle equal to a given one.*



Solution. Let ABCD be the trapezoid to be converted into another trapezoid, which has the

angle at A and the side AD common; but the other angle at D equal to the given angle *m*.

Convert the trapezoid ABCD into the triangle AED, having the same base AD and the angle at A common (see the rule given in problem XII); then this triangle again into a trapezoid ADBC, so that the angle ADG is equal to the given angle *m*; the trapezoid ADBC is the one sought.

* *

*

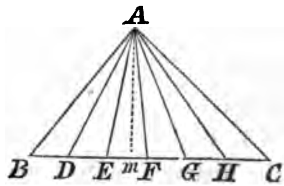
PART III.

Partition of figures by drawing.

PROBLEM XXXVI. *To divide a triangle from one of the vertices into a given number of parts.*

SOLUTION. Let ABC be the given triangle, which is to be divided, say, into six equal parts; let A be the vertex, from which the lines of division are to be drawn.

1. Divide the side BC, opposite the vertex A, into six equal parts, BD, DE, EF, FG, GH, HC.



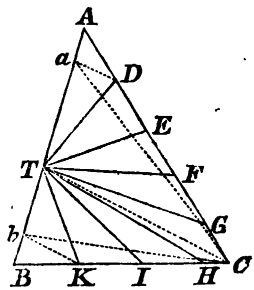
2. From A to the points of division D, E, F, G, H, draw the lines AD, AE, AF, AG, AH; the triangle ABC is

divided into the six equal triangles, ABD, ADE, AEF, AFG, AGH, AHC.

DEMON. The triangles ABD, ADE, AEF, AFG, AGH, AHC, are, in area, equal to one another, because they have equal bases and the same height A*m* (page 108).

Remark. If it is required to divide the triangle ABC according to a given proportion, it will only be necessary to divide the line BC in this proportion, and from A to draw lines to these points of division.

PROBLEM XXXVII. ***From a given point in one of the sides of a triangle, to divide it into a given number of equal parts.***



SOLUTION. Let ABC be the given triangle, which is to be divided into eight equal parts; the lines of division are to be drawn from T.

1. Make A*a* and B*b* equal to $\frac{1}{8}$ of AB, and from T draw the line TC to the vertex C of the triangle.

2. From *a* and *b* draw the lines *a* D, *b* K parallel to TC, meeting the sides AC, BC, in D and K.

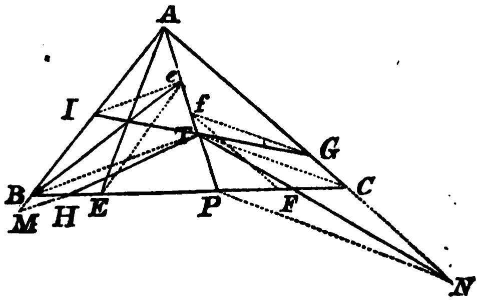
3. From AC, from A towards C, measure off the distance AD as many times as possible (in this case four times); and thus determine the points E, F, G; from BC, in the direction from B towards C, also measure off the distance BK

as many times as is possible, (here three times) and determine the points I, H.

4. From T draw the lines TD, TE, TF, TG, TH, TI, TK; then ATD, DTE, ETF, FTG, HTHC, HTI, ITK, KTB, are the eight equal parts of the triangle ABC.

Demon. Draw CA; then the triangle A *a* C is $\frac{1}{8}$ of the triangle ABC; because if AB is taken for the base of the triangle ABC, the base A *a* of the triangle A *a* C is, by construction, $\frac{1}{8}$ of the base of the triangle ABC, (see page 221). But the triangles *a* D C, is equal to the triangle *a* DT; because these two triangles are upon the same base *a* D and between the same parallels *a* D, TC; therefore (by adding to each of them the triangle *a* AD) the two triangles ADT and *a* AC are also equal; that is, ADT is also $\frac{1}{8}$ of the triangle ABC. In the same manner (by drawing the line *b* C), it may be proved that the triangle BKT is also $\frac{1}{8}$ of the triangle ABC. Further, the triangles ATD, DTE, ETF, FTG, are by construction all equal to one another (see the demonstration to the last problem); and for the same reason are the triangles BTK, KTI, ITH equal to one another; therefore each of the seven triangles ATD, DTE, ETF, FTG, BKT, KIT, ITH is $\frac{1}{8}$ of the triangle ABC; consequently the quadrilateral GTHC must be the remaining one eighth of the triangle ABC; and the area of the triangle ABC is divided into eight equal parts.

Problem XXXVIII. *To divide a triangle from a given point within it, into a given number of equal parts.*



Solution. Let ABC be the given triangle, which is to be divided, say, into five equal parts; T the point, from which the lines of division are to be drawn.

1. Through the point T and the vertex A of the triangle draw the line AT.

2. Take any side of the triangle, say BC, and make, when, as here, the triangle is to be divided into five equal parts, BE and CF equal to $\frac{1}{5}$ of BC, and draw the lines E *e*, F *f*, parallel to the sides AB, AC; these lines will meet the line AD in the points *e* and *f*.

3. From T draw the lines TB, TC to the vertices B and C of the triangle ABC, and from *e* and *f*, the lines *e* I, *f* G parallel to TB, TC.

4. Join TI, TG; then each of the triangles ATI, ATG, is $\frac{1}{5}$ of the given triangle ABC.

5. In order to determine the other points of division, it is only necessary to cut off from the sides AB, AC, as many distances equal to AI, AG, as is possible (see the solution of the last prob-

lem), and in the case where this can no longer be effected, or in which, as in the figure, this is impossible, proceed in the following manner:

a. Extend the two sides AB, BC, and then make IM equal to AI, and GN equal to AG.

b. From M and N draw the lines MH, NP parallel to BT and GT and determine thereby the points H and K.

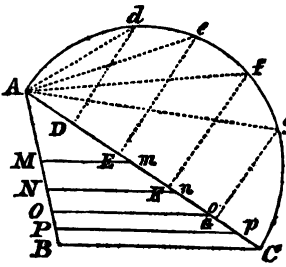
6. Draw TH, TP; each of the quadrilaterals BMHT, GNPT, is $\frac{1}{5}$ of the triangle ABC; consequently the triangle HTP is the remaining fifth of it. If HTP is not the last part, then it is merely necessary to divide this triangle (by the rule given in problem XXXVI) into as many equal parts as necessary.

Demon. Draw the auxiliary lines AE, B *e*; then the triangle ABE is one fifth of the triangle ABC; because BE is one fifth of the basis BC (problem XXXVI); further the triangle ABE is equal to the triangle AB *e*; because these two triangles are upon the same basis AB, and, by construction 2, between the same parallels AB, E *e*; and the last triangle AB *e* is also equal to the triangle ATI; because the triangle AB *e* consists of the two triangles AI *e* and I *e* B, which are equal to the two triangles AI *e* and IT *e*, (the two triangles IT *e* and I *e* B being upon the same base I *e*, and, by construction 3, between the same parallels I *e*, BT); therefore the triangle AIT is also one fifth of the triangle ABC; and in the same manner it can be proved that ATG is one fifth of the triangle ABC.

Further, the triangle GNT is equal to the triangle AGT (the basis GN being made equal to the basis AG, and the vertical point T being common to both the triangles); and the triangle GNT is equal to the quadrilateral CGPT; because the triangle CTN is equal to the triangle CTP (these two triangles being by construction upon the same base TC and between the same parallels TC, PN); therefore

the area of the quadrilateral CGPT is also one fifth of the triangle ABC; and in the same manner it may be proved that the area of the quadrilateral IBHT is one fifth of the triangle ABC; and as the two triangles AGT, AIT, together with the two quadrilaterals CGPT, IBHT, make together four fifths of the triangle ABC, the triangle HPT must be the remaining one fifth of it.

PROBLEM XXXIX. *To divide a given triangle into a given number of equal parts, and in such a way, that the lines of division may be parallel to a given side of the triangle.*



SOLUTION. Let ABC be the given triangle; let the number of the parts, into which it is required to be divided, be five, and BC the side, to which the lines of division are to be parallel.

1. Upon one of the other two sides, say AC, describe a semicircle, and divide the side AC into as many equal parts as the triangle is to be divided into; consequently, in the present case, into five; the points of section are D, E, F, G.

2. From these points of division draw the perpendiculars D *d*, E *e*, F *f*, G *g*, meeting the semicircle in the points *d*, *e*, *f*, *g*.

3. From A draw A d, A e, A f, A g; then make A m equal to A d, A n equal to A e, and so on, and by these means determine the points m, n, o, p.

4. From these points draw the lines mM, nN, oO, pP, parallel to the side BC; then AMm, MmNn, NnOo, OoPp, PpBC are the five equal parts of the triangle ABC, which were sought.

DEMON. Imagine the line dC drawn; the triangle AdC inscribed in the semicircle, is right-angular in d; consequently we have the proportion

$$AD : Ad = Ad : AC$$

and as in every geometrical proportion the product of the mean terms is equal to that of the extreme terms (principle 8 of 76 geom. prop. page), we have

$$Ad \times Ad = AD \times AC;$$

consequently also,

$$Am \times Am = AD \times AC;$$

(because A m is made equal to A d).

Further the triangles AM m, ABC are similar, because the line M m is drawn parallel to the side BC in the triangle ABC (page 82); and as the areas of similar triangles are to each other as the areas of the squares upon the corresponding sides (see page 124) we have the proportion

$$\text{triangle ABC} : \text{triangle AM}m = AC \times AC : Am \times Am;$$

therefore also

$$\text{triangle ABC} : \text{triangle AM}m = AC \times AC : AC \times AD,$$

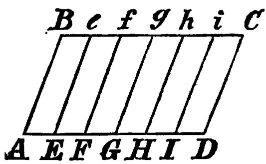
(because A m × A m is equal to AC × AD).

This last proportion expresses, that the area of the triangle ABC is as many times greater than the area of the triangle AM m, as AC times the side AC itself, is greater than AC times the side AD; consequently also, as AC is greater than AD (see the note to page 108). But the side AC, is, by construction 1, five times as great as AD, that is AD is one fifth of AC, therefore the area of

triangle AM m is also one fifth of the area of the triangle ABC. In like manner it may be proved, that the triangle AN n is two fifths of the triangle ABC; the triangle AO o three fifths, and the triangle AP p four fifths of it, from which the rest follows of course.

Remark. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will merely be necessary, as may be readily seen from the above, to divide the line AC according to this proportion, and then proceed as has been already shown.

PROBLEM XL. *To divide a parallelogram into a given number of equal parts in such a way, that the lines of division may be parallel to two opposite sides of the parallelogram.*

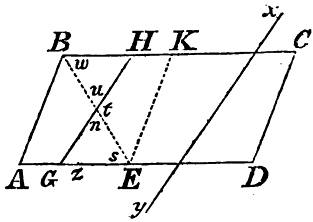


SOLUTION. Let ABCD be the parallelogram to be divided; let the number of parts be six; and let AB, CD, be the sides, to which the lines of division shall be parallel.

Divide one of the two other sides, say AD, into six equal parts, in E, F, G, H, I, and from these points draw the lines E e, F f, G g, H h, I i, parallel to the sides AB, CD; then the division is done.

Remark. If it is required to divide the parallelogram according to a given proportion, it will merely be necessary, instead of dividing the line AD into equal parts, to divide it according to the given proportion, and then proceed as before.

PROBLEM XLI. *To divide a parallelogram according to a given proportion by a line, which shall be parallel to a line given in position.*



SOLUTION. Let ABCD be the parallelogram to be divided.

1. Divide one of its sides, say AD, according to the given proportion; let the point of division be in z.

2. Make zE equal to the distance Az, and draw BE. Now if the line BE has the required position, the triangle ABE and the quadrilateral BCDE are the parts sought.

3. But if the line of division is required to be parallel to the line xy, bisect the line BE in n, and through this point draw the line GH parallel to xy, then the two quadrilaterals ABHG, HCDG will be the required parts.

DEMON. Draw EK parallel to AB. Then the two parallelograms ABEK, ABCD, having the same height, their areas are in the same ratio as their bases AE, AD (see page 108, 6th); that is, we have the proportion

parall. ABEK : parall. ABCD = AE : AD;

therefore

$\frac{1}{2}$ of parall. ABEK : parall. ABCD = $\frac{1}{2}$ of AE : AD,

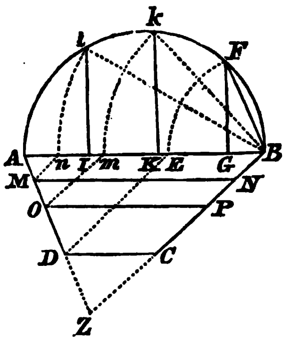
and because the triangle AEB is equal to half the parallelogram ABEK, and half of AE is, by construction 2, equal to A z, we have

triangle AEB : parall. ABCD = A z : AD.

This last proportion expresses that the area of the parallelogram ABCD is as many times greater than the area of the triangle ABE, as the line AD is greater than A z; consequently if BE has the required position, the triangle ABE is one of the required parts, and therefore the trapezoid BEDC the other.

Further, the line BE is, (by construction 3) bisected; the angles u and n are opposite angles at the vertex, (page 24), and w and s are alternate angles (page 33, Ist); therefore the triangle B t H, having the side B t, and the two adjacent angles w and u, equal to the side t E, and the two adjacent angles n and s in the triangle G t E, these two triangles are equal to one another; consequently the area of the trapezoid ABGH (composed of the quadrilateral ABG t and the triangle B t H) is equal to the area of the triangle ABE, (composed of the same quadrilateral and the triangle G t E), which proves the correctness of construction 3.

PROBLEM XLII. *To divide a trapezoid into a given number of equal parts so that the lines of division may be parallel to the parallel sides of that trapezoid.*



Solution. Let ABCD be the given trapezoid, which is to be divided into three equal parts.

1. Upon AB, the greater of the two parallel sides, describe a semicircle; draw DE parallel to CB; and from B, with the radius BE, describe the arc of a circle EF, cutting the semicircle in F.

2. From F draw FG perpendicular to AB, and divide the line AG into three equal parts in K and I; and from these points draw the perpendiculars K *k*, I *i*.

3. Upon AB, from B towards A, take the distances B *m*, B *n*, equal to B *k*, B *i*; from these points draw the lines *m* O, *n* M, parallel to BC; and from the points O, M, in which these parallels meet the side AD, draw the lines MN, OP, parallel to AB; then ABNM, MNPO

OPCD, are the three required parts of the trapezoid ABCD.

Demon. Extend the lines AD, BC, until they meet in Z. Then the triangles DCZ, OPZ, MNZ, ABZ are all similar to each other; because DC, OP, MN, AB, are parallel, (see page 82.) Further, we have, by construction 3,

DC equal to BE and to BF
OP " " B m " " B k
MN " " B n " " B i.

The areas of the two similar triangles OPZ, CDZ, are in the ratio of the squares upon the corresponding sides, that is, we have the proportion

triangle OPZ : triangle DCZ = OP $\times$ OP : CD $\times$ CD,

and since OP is equal B k, and CD to BF, also

triangle OPZ : triangle DCZ = B k $\times$ B k : BF $\times$ BF.

Imagine AF and FB joined; the triangle AFB would be right-angular in F, and we should have the proportion

BG : BF = BF : AB. (See the demon. to problem XXXIX.)

and for the same reason we have

BK : B k = B k : AB.

Taking the product of the mean and extreme terms of the two last proportions, we have

BG $\times$ AB equal to BF $\times$ BF, and
BK $\times$ AB " " B k $\times$ B k (principle 8th of prop. page 76);

therefore by writing in the above proportion, BG $\times$ AB instead of BF $\times$ BF (its equal), and BK $\times$ AB instead of B k $\times$ B k, we have

triangle OPZ : triangle DCZ = AB $\times$ BK : AB $\times$ BG,

whence we infer (as in the demon. to problem XXXIX), that

triangle OPZ : triangle DCZ = BK : BG ;

consequently also

triangle OPZ — triangle DCZ : triangle DCZ = BK — BG : GB;

which is read thus:

triangle OPZ less the triangle DPZ, is to the triangle DCZ, as the line BK less the line BG, is to the line BG;

that is, trapezoid DOPC : triangle DCZ = GK : BG,

and as GK is, by construction 2, equal to $\frac{1}{3}$ of AG,

trapezoid DOPC : triangle DCZ = $\frac{1}{3}$ AG : BG.

In like manner it may be proved that

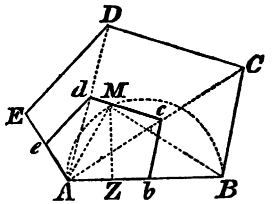
trapezoid DMNC : triangle DCZ = $\frac{2}{3}$ AG : BG and

trapezoid DABC : triangle DCZ = AG : BG.

These proportions express that the three trapezoids DOPC, DMNC, DABC, are to each other in the same proportion as one third is to two thirds to three thirds; or, which is the same, as one is to two, to three; whence the rest of the demonstration follows of course.

Remark. If it is required not to divide the trapezoid ABCD into equal parts, but according to a given proportion, it will only be necessary to divide the line AG in this proportion, and then proceed as before.

PROBLEM XLIII. ***To divide a given figure into two parts according to a given proportion, and in such a way, that one of the parts may be similar to the whole figure.***



SOLUTION. Let ABCDE be the given figure.

1. Divide one side of the figure, say AB, according to the given proportion; let the point of division be in Z.

2. Upon AB, as a diameter, describe a semicircle, and from Z draw the perpendicular ZM, meeting the semicircle in M.

3. Make $Ab = AM$, and upon Ab describe a figure $Abcde$, which is similar to the given one ABCDE (see problem XXXIII); the line $bcde$ divides the figure in the manner required.

DEMON. The areas of the two similar figures $Abcde$, ABCDE, are to each other, as the squares upon their corresponding sides (page 125); therefore we have the proportion,

$$ABCDE : Abcde = AB \times AB : Ab \times Ab.$$

Draw AM and BM; then AM is a mean proportional between AZ, and AB; that is, we have

$$AZ : AM = AM : AB,$$

and as Ab is by construction equal to AM,

$$AZ : Ab = Ab : AB;$$

consequently the product $Ab \times Ab$ is equal to $AZ \times AB$.

Writing $AZ \times AB$ instead of $Ab \times Ab$ (its equal) in the first proportion, we have

$$ABCDE : Abcde = AB \times AB : AB \times AZ,$$

hence $ABCDE : Abcde = AB : AZ$; and therefore

$$ABCDE - Abcde : Abcde = AB - AZ : AZ;$$

which is read thus:

ABCDE less $Abcde$ is to $Abcde$, as AB less AZ is to AZ; that is,

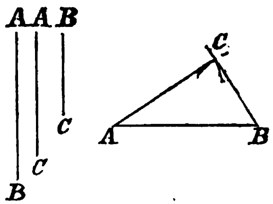
BCDE $edcb$ is to $Abcde$, as ZB is to AZ;

consequently the figure ABCDE is divided according to the given proportion, in which the line AB is divided.

PART IV.

Construction of triangles.

PROBLEM XLIV. *The three sides of a triangle being given, to construct the triangle.*



SOLUTION. Let AB, AC, BC, be the three given sides of the triangle.

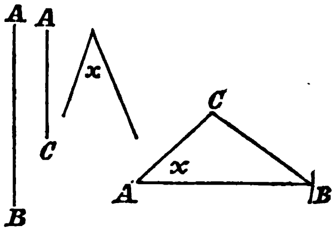
1. Take any side, say AB, and from A as a centre, with the radius AC, describe an arc of a circle.

2. From B, as a centre, with the radius BC, describe another arc, cutting the first.

3. From the point of intersection C, draw the straight lines CA, CB; the triangle ABC is the one required.

The demonstration follows immediately from Query 4th Sect. II.

PROBLEM XLV. *Two sides and the angle included by them being given, to construct the triangle.*



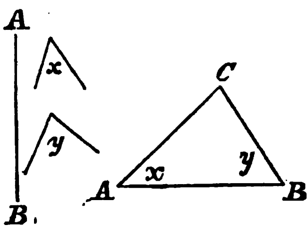
SOLUTION. Let AB, AC, be the two given sides, and x the angle included by them.

1. Construct an angle equal to the angle x, (problem VI); make one of the legs equal to the side AB, and the other to the side AC.

2. Join BC; the triangle ABC is the one required.

The demonstration follows from Query 1, Sect. II.

PROBLEM XLVI. *One side and the two adjacent angles being given, to construct the triangle.*



SOLUTION. Let AB be the given side, and x and y, the two adjacent angles.

1. At the two extremities of the line AB, construct the angles x and y, and extend their legs AC, BC, until they meet in the point C; the triangle ABC is the one required.

The demonstration follows from Query 2, Sect. II.

PROBLEM XLVII. *Two sides and the angle opposite to the greater of them being given, to construct the triangle.*

SOLUTION. Let AC, BC (see the figure to problem XLV) be the two given sides, and x the angle, which is opposite to the greater of them (the side BC).

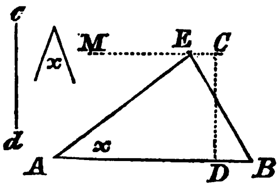
1. Upon an indefinite straight line construct an angle equal to the angle x.

2. Make the leg AC of this angle equal to the smaller side AC, and from C as a centre, with the radius CB equal to the greater side, describe an arc of a circle, cutting the line AB in the point B.

3. Join BC; the triangle ABC is the one required.

The demonstration follows from Query 10th, Sect. II.

PROBLEM XLVIII. *The basis of a triangle, one of the adjacent angles and the height being given, to construct the triangle.*



SOLUTION. Let AB be the given basis, x one of the adjacent angles, and $c\,d$ the height.

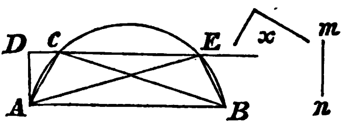
1. In any point of the line AB, draw a perpendicular CD equal to $c\,d$; and through C a line parallel to AB.

2. In A make an angle equal to the given angle x, and extend the leg AE until it meets the line MN.

3. Join EB; the triangle AEB is the one required.

The demonstration is sufficiently evident from the construction.

PROBLEM XLIX. *The basis, the angle opposite to it, and the height of a triangle being given, to construct the triangle.*



SOLUTION. Let AB be the given base, x the angle opposite to it, and $m\,n$ the height of the triangle.

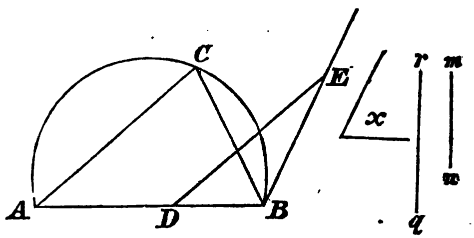
1. Upon the base AB describe a segment of a circle containing a given angle x (see problem XVI).

2. In A draw a perpendicular AD equal to the given height $m\,n$, and through D draw DE parallel to AB.

3. From C and E, where this parallel cuts the segment, draw the straight lines CA, CB, EA, EB; either of the two triangles ACB, AEB will be the one required.

The demonstration follows from the construction.

PROBLEM L. *The basis of a triangle, the angle opposite to it, and the ratio of the two other sides being given, to construct the triangle.*



SOLUTION. Let AB be the given base, x the angle opposite to it; and let the two remaining sides bear to each other the same ratio, which exists between the two lines $m\,n$ and $p\,q$.

1. Upon AB describe a segment of a circle capable of the given angle x (see problem XVI).

2. In B make an angle ABE equal to the angle x; make BE equal to the line $r\,q$, BD equal to $m\,n$, and join DE.

3. From A draw the line AC parallel to DE, and from the point C, where it meets the segment, draw the line CB; the triangle ABC is the one required.

DEMON. The triangle ABC is similar to the triangle DBE; because the two angles CAB and ACB in the one are equal to the two angles BDE, DBE, in the other, each to each* (page 87, 1st); therefore we have the proportion

$$AC : BC = BE : BD,$$

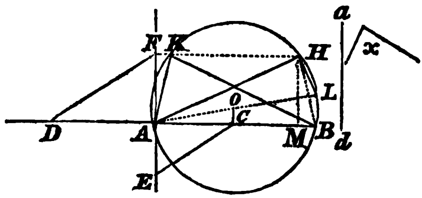
which expresses that the two sides AC, BC of the triangle are in the same ratio as the sides EB, BD, of the triangle BDE; consequently they are also as the lines $m\,n$, $p\,q$; because BE and BD are, by construction, equal to $m\,n$, $p\,q$. The rest of the demonstration is evident from the construction.

PROBLEM LI. *The basis of a triangle, the angle opposite to it, and the square, which, in area, is equal to the rectangle of the two remaining sides, being given, to construct the triangle.*†

---

* CAB and EDB being alternate angles, and each of the angles ACB, DCB, being made equal to the given angle x.

† By the rectangle of the two remaining sides is meant a rectangle, whose base is one of these sides, and whose height is the other.



SOLUTION. Let AB be the given base, x the angle opposite to it, and $a\,d$ the side of the square, equal to the rectangle of the two remaining sides.

1. Upon AB construct the segment AKHB of a circle, capable of the given angle x.

2. Extend AB towards D, and in A draw the perpendicular AF.

3. Make AC equal to the radius AO, AD to the side $a\,d$ of the given square, and AE to half of AD; join EC, and from D draw DF parallel to EC.

4. Through the point F, where this parallel meets the perpendicular, draw FH parallel to AB; and from the points K and H, where this meets the segment, the lines AK, KB, AH, HB; then either of the two triangles AKB, AHB is the one required.

DEMON. Draw the diameter AL, and from either of the points K, H, say H, drop the perpendicular HM upon AB. The triangle ALH is similar to the triangle MBH; for the triangle ALH being inscribed in a semicircle, each of these triangles is right-angular; and the two angles ALH, ABH, are equal; because

each of them is an angle at the circumference, with its legs standing on the extremities of the same arc AKH (page 142, 2d); therefore the remaining angles HAL and MHB are also equal (page 87, 1st); and the corresponding sides of the two triangles ALH, MBH are in the geometrical proportion

$$AH : AL = HM : HB;$$

consequently we have

$AL \times HM = AH \times HB$ (princ. 8th of geo. prop. page 76).

This proportion expresses, that the area of the rectangle, which has for its base the diameter AL, and its height equal to that of the triangle AHB, is equal to the area of the rectangle, which has the side AH for its base and the side HB for its height.* Further, it is easy to perceive that from the similar triangles ACE, ADF, we have the proportion

$$AD : AF = AC : AE;$$

consequently also,

$$AD : AF = 2\,AC : 2\,AE;$$

therefore,

$$2\,AC \times AF = 2\,AE \times AD; \text{ or}$$

$$\text{diam. } AL \times AF = a\,d \times a\,d,$$

(because AD is made equal to *a d*, AE to half of AD, and AC to the radius A *o*, of the circle).

This proportion expresses that the area of the square upon *a d* is equal to that of the rectangle, whose base is the diameter AL, and whose height AF is equal to MH (see the figure); therefore the area of the rectangle upon the basis AM, and of the height HB being equal to the area of the rectangle upon the diameter AL and of the height HM, is also equal to that of the square upon *a d;* and the same may be proved of the rectangle of the two sides AK, KB, of the triangle AKB.

The rest of the demonstration is sufficiently evident from the construction.

---

* For the area of a rectangle is found by multiplying the base by the height.

For the sake of practice the learner may now solve the following

PROBLEMS:

1. The basis of a triangle, the angle opposite to it, and the sum of the two remaining sides being given, to construct the triangle.

2. The basis of a triangle, the angle opposite to it, and a line equal to the difference of the two remaining sides being given, to construct the triangle.

3. The basis of a triangle, the angle opposite to it, and a side of the square, which is equal to the sum of the squares upon the two remaining sides, being given, to construct the triangle.

4. The basis of a triangle, the angle opposite to it, and a side of the square, which is equal to the difference between the squares upon the two remaining sides, being given, to construct the triangle.

5. The basis of a triangle, the angle opposite to it, and the sum of the height and the two remaining sides, being given, to construct the triangle.

6. The basis of a triangle, the angle opposite to it, and the sum of the height and difference between the two remaining sides, being given, to construct the triangle.

7. The basis of a triangle, the angle opposite to it, and the difference between the sum of the two remaining sides and the height, being given, to construct the triangle.

8. The basis of a triangle, the angle opposite to it, and the difference between the difference of the two remaining sides, and the height of the triangle, being given, to construct the triangle.

9. The basis of a triangle, the angle opposite to it, and the ratio of the sum of the two remaining sides to the height of the triangle being given, to construct the triangle.

10. The basis of a triangle, the angle opposite to it, and the ratio of the difference between the two remaining sides to the height of the triangle, being given, to construct the triangle.

11. The basis, height, and sum of the two remaining sides of a triangle being given, to construct the triangle.

12. The basis, height, and difference between the two remaining sides of a triangle being given, to construct the triangle.

13. The basis and height of a triangle, together with the ratio of the two remaining sides, being given, to construct the triangle.

14. The basis and height of a triangle, and a square equal to the rectangle of the two remaining sides, being given, to construct the triangle.

15. The basis and height of a triangle, and a square which is equal to the sum of the squares upon the two remaining sides, being given, to construct the triangle.

16. The basis and height of a triangle, and a square which is equal to the difference between the squares upon the two remaining sides, being given, to construct the triangle.

17. The angle opposite to the basis of a triangle, the sum of the basis and height, and that of the two remaining sides, being given, to construct the triangle.

18. The angle opposite to the basis of a triangle, the sum of its basis and height, and the difference of the two remaining sides of the triangle, being given, to construct the triangle.

19. The angle opposite to the basis of a triangle, the sum of its basis and height, and the ratio of the two remaining sides, being given, to construct the triangle.

20. The angle opposite to the basis of a triangle, the sum of its basis and height, and the square equal to the rectangle of the two remaining sides, being given, to construct the triangle.

21. The angle opposite to the basis, the sum of the basis and height, and the sum of the three sides, being given, to construct the triangle.

22. The angle opposite to the basis, the sum of its basis and height, and the difference between the sum of the two remaining sides and the basis, being given, to construct the triangle.

23. The angle opposite to the basis, the sum of its basis and height, and the difference between the basis and the difference of the two remaining sides, being given, to construct the triangle.

24. The angle opposite to the basis, the sum of the basis and height, and the ratio of the sum of the two remaining sides to the basis, being given, to construct the triangle.

25. The angle opposite to the basis, the sum of the basis and height, and the ratio of the difference between the two remaining sides to the basis, being given, to construct the triangle.

26. The angle opposite to the basis, the height, and the square, which is equal to the sum of the squares upon the two remaining sides, being given, to construct the triangle.

27. The angle opposite the basis, the height and the sum of the three sides, being given, to construct the triangle.

28. The angle opposite to the basis, the height, and the radius of the circumscribed circle, being given, to construct the triangle.

29. The angle opposite to the basis, the radius of the inscribed circle, and the sum of the three sides, being given, to construct the triangle.

30. The angle opposite to the basis, the radius of the inscribed circle, and the ratio of the sum of the two remaining sides to the basis, being given, to construct the triangle.

31. The angle opposite to the basis, the radius of the inscribed circle, and the square whose area is equal to that of the rectangle of the sum of the two remaining sides and the basis, being given, to construct the triangle.

32. The angle opposite to the basis, the radius of the inscribed circle, and the square whose area is equal to the difference between the sum of the squares of the two remaining sides and the square upon the basis, being given, to construct the triangle.

33. The basis, the line drawn from the vertex of the opposite angle to the middle of the basis, and the sum of the two remaining sides, being given, to construct the triangle.

34. The basis, the line drawn from the vertex of the opposite angle to the middle of the basis, and the difference between the two remaining sides, being given, to construct the triangle.

35. The perpendiculars, dropped from the three vertices upon the opposite sides, being given, to construct the triangle.

36. The lines drawn from the three vertices to the middle of the opposite sides being given, to construct the triangle.

37. The lines drawn from two vertices to the two opposite sides, and the sum of these sides being given, to construct the triangle.

38. Two sides, and the line drawn from the vertex of the included angle to the middle of the opposite sides, being given, to construct the triangle.

39. To construct a right-angular triangle, the difference between the hypothenuse and one of the sides of which, is equal to the difference between that and the remaining sides, and the area is equal to a given square.

40. To construct a right-angular triangle, one of the sides of which is a mean proportional between the hypothenuse and the other side, and the area is equal to a given square.

41. To place a given triangle upon another so that the vertices of the latter shall fall in the sides of the former.

42. In a given triangle to construct a rectangle, whose area is equal to a given square.

43. In a given triangle, to construct a rectangle, the sum of the sides of which is equal to a given line.

44. In a given triangle to construct a rectangle, the difference between two adjacent sides of which is equal to a given line.

45. In a given triangle to construct a rectangle, whose diagonal is equal to a given line

46. In a parallelogram to construct a square, whose vertices are in the sides of the parallelogram.

47. In a given quadrilateral to construct a parallelogram, whose sides are parallel to two given straight lines.

48. The difference between one of the diagonals and one of the sides of the square being given, to construct the square.

49. The three angles and the sum of the three sides of a triangle being given, to construct the triangle.

50. To describe a circle, which shall pass through two given points, and touch a given straight line.

51. To describe a circle, which shall pass through a given point, and touch two given straight lines.

52. To describe a circle, which shall pass through two given points, and touch a given circle.

53. To describe a circle, which shall pass through a given point, and touch two given circles.

54. To describe a circle, which shall touch three given circles.

55. To describe a circle, which shall pass through a given point, touch a given straight line, and also a given circle.

56. To describe a circle, which shall touch two given straight lines and also a given circle.

57. To describe a circle, which shall touch a given straight line and two given circles.

58. To construct a rectangle, the ratio of two adjacent sides of which is given, and whose area shall diminish by the area of a given square, if the basis and height diminish by given straight lines.

59. To construct a rectangle, the ratio of two adjacent sides of which is given, and whose area increases by that of a given square, if the basis and height increase by given straight lines.

60. To find in lines, the ratio of two given parallelograms.

61. Several similar figures being given, to construct a figure which is similar to each of them and equal to their sum.

62. From a given point in one of the sides of a triangle, to divide it in a given ratio.

63. From a given point within a triangle, to divide it in a given ratio.

64. To divide a triangle into a given number of equal parts, in such a way that the points of division shall be parallel to a straight line, given in position.

APPENDIX.

Containing Exercises for the Slate.

1. The side of a square being 12 feet, what is its area?

2. What if the side is 12 rods, miles, &c.?

3. What is the side of a square, whose area is one square foot?

4. What that of a square, whose area is one square yard, rod, mile, &c.?

5. What that of a square of 4, 9, 16, 25, 36, 49, 64, 81, 100 feet?

6. What is the area of a rectangle, whose base is 50 feet 3 inches, and whose height 10 feet 4 inches 2 seconds?

7. What that of a rectangle whose base is 40 feet 3 inches 9 seconds, and whose height is 12½ feet?

8. If the area of a rectangle is 240 square feet 19 square inches, and its basis measures 30 feet, what is its height?

9. What is the height of a rectangle, whose base is 10 feet and whose area is 40 square feet?

10. What is the height of a rectangle, whose basis is 4 feet, and whose height is 3 inches 5 seconds?

11. What is the area of a parallelogram of 10 feet basis and 3 feet 4 inches high?

12. The height of a parallelogram is 5 feet, and the area 40 square feet, what is its basis?

13. The sum of the two parallel sides of a trapezoid is 12 feet and their distance 4 inches, what is the area of the trapezoid?

14. The area of a trapezoid is 24 square feet, and its height is 4 inches 3 seconds, what is its basis?

15. What is the difference between a triangle, whose basis is 10 feet 3 inches and height 9 feet, and a triangle of 3 feet basis and 11 inches height?

16. What is the difference between a trapezoid, the sum of the two parallel sides of which is 14 feet 3 inches and height 9 inches, and a square upon 9 inches?

17. What is the sum of the areas of a triangle of 3 feet basis and 9 inches height, a square upon 14 feet 3 inches, and a rectangle whose basis is 3 feet 2 inches 9 seconds, and height 4 inches 5 seconds?

18. What is the area of a circle, whose radius is 9 inches.

19. What that of a circle whose radius is 10 feet?

20. What that of a circle, whose radius is 9 feet 6 inches 2 seconds?

21. The area of a circle is 240 square feet, what is its radius or diameter? *

22. The area of a circle is 14 square feet 13 square inches, its radius is 5 feet 3 inches, what is its circumference?

23. What is the length of an arc of 14 degrees 29 minutes 24 seconds, in a circle whose radius is 14 inches?

24. What, that of an arc of 6 degrees 9 seconds, in a circle whose radius is 1 foot?

25. What, that of an arc of 9 seconds, in a circle whose radius is 1 inch?

26. What is the area of a sector of 15 degrees, in a circle whose radius is 3 feet?

27. What, that of a sector of 19 degrees 45 minutes, in a circle whose radius is 1 foot 3 inches?

The teacher may now vary and multiply these questions.

*
*   *

END.

---

* Divide the area by the circumference, and extract the square root of the quotient, the answer is the radius of the circle.

Note to Principle 5th of Geometrical Proportions, page 74.

For the same reason, that the second term of a geometrical proportion may be *added* once or any number of times to the first term, and the fourth term, the same number of times to the third term, without destroying the proportion, the second term may also be *subtracted* once or any number of times from the first term, provided the fourth term be the same number of times subtracted from the third term, and the result will still be a geometrical proportion.

If, in the geometrical proportion

$$AB : a\,b = AC : a\,c,$$

the first term (AB) is twice as great as $a\,b$, and AC twice as great as $a\,c$, we shall, by subtracting $a\,b$ from AB, and $a\,c$ from AC, make the two terms in each ratio equal; that is, we shall have a new proportion

$$AB - a\,b : a\,b = AC - a\,c : ac.$$

If AB were three times as great as $a\,b$, AC would, of course, be three times as great as $a\,c$; and therefore, by subtracting $a\,b$ from AB, and $a\,c$ from AC, the first term (AB — ab) in the last proportion would still be twice as great as $a\,b$; and for the same reason would AC — $a\,c$, be twice as great as $a\,c$. If AB were four times as great as $a\,b$, AC would be four times as great as $a\,c$, and therefore by subtracting $a\,b$ from AB, and $a\,c$ from AC, the first term (AB — $a\,b$) in the last proportion would be three times as great as the second term $a\,b$, and for the same reason would AC — ac be three times as great as $a\,c$. In the same manner this principle may be applied to every other geometrical proportion; and it may also be proved that the first term of a geometrical proportion may be once or any number of times subtracted from the second term, provided the third term is the same number of times subtracted from the fourth term, without destroying the proportion.

ERRATA.

Page 9, in the figure read 'quadrant' instead of 'quadrat.'

	ibid.	line	15, omit 'it.'
"	12	"	27, read 'parallel' instead of 'equal.'
"	16	"	8, read 'vertices' instead of 'verteces.'
"	ibid.	"	13, read 'sometimes' instead of 'something.'
"	17	"	2, read 'vertices' instead of 'verteces.'
"	34	"	1, read 'learned' instead of 'learnt.'
"	61	"	13, read 'diagonals' instead of 'diagonal.'
"	85	"	7, read 'Query 16' instead of 'Query 15.'
"	89	"	3, read 'condition 4th' instead of 'condition 3d.'
"	117	"	20, read 'three' instead of 'two.'
"	ibid.	"	21, read '19' instead of '18.'
"	"	"	22, read '7' instead of '6.'
"	123	"	1, read 'BCM' instead of 'CBM.'
"	126	"	3, read 'or the areas of the squares upon, &c.' instead of 'are to the areas of, &c.'
"	132	"	12, read 'exteriorly' instead of 'interiorly.'
"	176	"	3, read 'of tangent;' instead of 'of a tangent.'
"	186	"	22, read '*a*' instead of 'A.'
"	190	"	15, omit '*a d.*'
"	204	"	17, read 'ABE' instead of 'ABC.'
"	214	"	4, read 'HDG' instead of 'ADG.'
"	223	"	5, read 'HTGC' instead of 'HTHC.'
"	225	"	7, read 'CT' instead of 'GT.'

It is hoped that *all* the errors are carefully enumerated here.'